SLOVAK UNIVERSITY OF TECHNOLOGY IN BRATISLAVA
FACULTY OF CHEMICAL AND FOOD TECHNOLOGY

A REDESIGN OF STABILIZING CONDITIONS FOR MODEL PREDICTIVE CONTROL

DIPLOMA THESIS

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2010 **Bc. Soňa Gibaštíková**

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Study programme:	Automation and Informatization in Chemistry and Food Industry
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Workplace:	TU Eindhoven
Supervisor:	Ing. Michal Kvasnica PhD.
Co-supervisor:	Dr. Mircea Lazar

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Topic: **A Redesign of Stabilizing Conditions for Model Predictive Control**

Specification of Assignment:

The usual way to guarantee stability of model predictive control strategies is based on a terminal cost function and terminal. Objective of this work is to analyze the closed loop stability problem of constrained linear systems and presents an approach when the terminal constraints are removed. This stabilizing method requires a non- empty set of optimal terminal controls which ensure the decreasing of cost function and new stabilizing condition is established by weighting the terminal cost. The shortest stabilizing and feasible horizon for given system can be determined very easily in explicit way (i.e. the intersection between set of optimal terminal controls and input constraint set is non- empty). Prediction horizon is shorter then in method with terminal constraint and this approach leads to reduction of computational complexity of optimization problem. Based on these results, practical procedures to calculate mentioned set of optimal terminal controls for SISO and MIMO systems are to be presented presented and illustrated on practical examples.

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A special thanks goes to my family, my friends and boyfriend for all their support and love.

DECLARATION ON WORD OF HONOUR

I hereby declare, that the research and results presented in this thesis were conducted by me using the mentioned literature and advices of my supervisors.

In Bratislava, 20 May 2010 ……...

signature

ABSTRACT

The usual way to guarantee stability of model predictive control strategies is based on a terminal cost function and terminal set. This thesis analyzes the closed loop stability problem of constrained linear systems and presents an approach when the terminal constraints are removed. This stabilizing method requires a non- empty set of stabilizing terminal controls which ensure the decreasing of cost function and new stabilizing condition is established by weighting the terminal cost. The shortest stabilizing and feasible horizon for given system can be determined very easily in explicit way (i.e. the intersection between set of stabilizing terminal controls and input constraint set has to be non- empty). Prediction horizon is shorter then in method with terminal constraints and this approach leads to reduction of computational complexity of optimization problem. Based on these results, practical procedures to calculate mentioned set of stabilizing terminal controls for SISO and MIMO systems are presented. Moreover, some extensions to infinity norm are published. Finally, the performance of predictive controller is illustrated on some examples and some comparisons are made.

Keywords: Model predictive control, Stability, Feasibility, Constraints, SISO and MIMO systems, Infinity norm

SÚHRN

Štandardný spôsob garancie stability v stratégiách prediktívneho riadenia je založený na koncovej penalizačnej funkcii a koncovom ohraničení stavu. Táto práca analyzuje problém stability uzavretého regulačného obvodu lineárnych systémov s ohraničeniami a navrhuje metódu bez ohraničení koncového stavu. Tento prístup vyžaduje neprázdnu množinu stabilizujúcich koncových vstupov, ktoré zaručujú klesanie účelovej funkcie a zároveň je zavedená nová stabilizujúca podmienka pomocou váhovania koncovej penalizačnej matice. Najkratší stabilizujúci a realizovateľný predikčný horizont pre daný systém je stanovený jednoduchým explicitným spôsobom (prienik medzi množinou stabilizujúcich koncových vstupov a množinou vstupných ohraničení musí byť neprázdny). Predikčný horizont je kratší v porovnaní s metódou s koncovými ohraničeniami stavov a tento prístup vedie k redukcii výpočtovej náročnosti optimalizačného problému. Na základe týchto výsledkov sú prezentované postupy na výpočet spomínaných stabilizujúcich množín koncových vstupov pre SISO a MIMO systémy. V práci je publikované aj rozšírenie danej metódy na problémy pracujúce s nekonečno normami. Nakoniec sú zobrazené výsledky prediktívneho riadenia na niekoľkých príkladoch a výhody prístupu sú demonštrované pomocou názorného porovnania s metódou koncového ohraničenia stavu.

Kľúčové slová: Prediktívne riadenie, Stabilita, Realizovateľnosť, Ohraničenia, SISO a MIMO systémy, Nekonečno norma

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INTRODUCTION

This thesis analyzes the closed loop stability of constrained linear systems when model predictive control is employed. The usual way to guarantee stability of model predictive control is based on a terminal cost function and terminal set constraints. Terminal constraints are removed and proposed approach belongs to the terminal cost function framework. The terminal cost is calculated to satisfy that set of the terminal controls is nonempty. The set of stabilizing terminal controls is defined as set of all terminal inputs which guarantee decreasing of cost function for finite horizon. This ensures stability in terms of Lyapunov for any prediction horizon. The approach yields the same results as any other terminal cost or terminal equality constraints method for unconstrained linear systems. Nearly almost application imposes constraints. States and inputs constraints are limited by physical realization, stability and efficiency. Because of the set of stabilizing terminal controls is a function of terminal state x_{k+N} (i.e. function of prediction horizon), we can for the particular prediction horizon determine feasibility. The horizon is feasible if the intersection between the set of stabilizing controls and the set of input constraints is non empty. The biggest advantage of this approach is absence of terminal constraints set what makes the problem much easier to solve and computational burden is reduced. Also constructing of terminal sets is very difficult many times so another problem is deleted.

The layout of the thesis is as follows. Section 2 contains the problem formulation and Section 3 gives stability analysis. The main results, concerning a stabilizing method for constrained SISO and MIMO systems, are discussed in Section 4. Finally, the illustrative examples are presented in Section 5.

1 MPC

Control design methods based on MPC concept have found wide acceptance, both in industrial applications area and in the academic world. Many reports of industrial applications confirm its popularity. The development of modern control concepts can be tracked back to the work of *Calman (1960s)* who studied linear quadratic regulator (LQR) designed to minimize a quadratic cost function without any constraints. In the late 70s *Rachael's "Model Predictive Heuristic Control"* and *"Dynamic Matrix control"* from *Shell engineers* described successful application of MPC in industry. The receding horizon principle, the core of all MPC algorithms, was proposed by *Propoi (1963)*. Later the next generation of MPC such as *"Quadratic Dynamic Matrix Control" (Garcia, Mrschedi, 1986)* used quadratic programming to solve constrained optimal open loop problem. Another line of work arose independently around adaptive control ideas *(Peterka 1984, Clarke 1987)*. Next the extension to nonlinear systems followed shortly. All the methods of MPC before 2000 were based on on-line optimization. *Bemporad et al.* and *Tondel et al.* derived explicit solutions to the constrained MPC problem which could increase the area of use where MPC has not traditionally been used because of the need for high sampling rates or software issues. Obviously constraints are presented in all control systems due to physical, environmental and economic limits on plant operation and in this context classical control methods are not sufficient to ensure good performance comparing to MPC. Model predictive control technology is a control standard in chemical and hydrocarbon processing industries. MPC is also widely used in other industries including paper, power, food processing, automotive, metallurgy and pharmaceuticals.

To sum up, model predictive control (MPC) is a form of control in which current control action is obtained by solving at which sampling time open loop optimal control problem over finite horizon and only a first element of the optimal control sequence is applied to the plant and the rest is discarded. The plant state is sampled again and optimization is repeated starting from the new state yielding a new control sequence. The prediction horizon keeps being shifted forward and for this reason MPC is also referred to as *the receding horizon strategy*. It is the same like chess player who develops strategy

for next steps, implement the next move and then makes another prediction of situation after observing how the opponent has moved. Future actions are calculated by minimizing cost function subject to variable constraints on the prediction horizon. All MPC schemes are using the principles mentioned above and the difference is only in the type of model that is utilized (linear, nonlinear and hybrid models) and online or off-line computation.

1.2 Problem formulation

Consider the following discrete time linear system defined by

$$x_{k+1} = Ax_k + Bu_k \tag{1}$$

where $x_k \in R^n$ represents the system states and $u_k \in R^m$ input signals and $A \in R^{nxn}, B \in R^{nxm}$ are the matrices of the system. It is assumed that the states and inputs are subjects to constraints:

$$x_k \in X \subseteq R^n \tag{2.1}$$

$$u_k \in U \subseteq R^m \tag{2.2}$$

where these constraints should hold $\forall k \geq 0$ and X, U are convex and compact sets containing the origin.

For state x_k the cost function is defined as classic quadratic cost function with a terminal cost.

$$J(x_k, \mathbf{u_k}) = x_{k+N}^T P x_{k+N} + \sum_{i=0}^{N-1} \left(x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i} \right) \tag{3}$$

where N is the length of prediction horizon (equal to control horizon), Q, R, P are symmetric positive definite state, control and terminal weighting matrices and

$$\mathbf{u_k} = \{u_k, u_{k+1}, ..., u_{k+N-1}\} \tag{4}$$

is a sequence of control actions to be applied to the system at current state x_k.

The optimal control sequence is obtained by solving, at each step, the following open-loop optimization problem

$$J^*(x_k) = \min_{\mathbf{u_k}} J(x_k, \mathbf{u_k}) = x_{k+N}^T P x_{k+n} + \sum_{i=0}^{N-1} \left(x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i} \right) \tag{5}$$

s.t. $$x_{k+1} = Ax_k + Bu_k \tag{6.1}$$

$$x_k \in X, \forall k = 1, ..., N \tag{6.2}$$

$$u_k \in U, \forall k = 0, ..., N-1 \tag{6.3}$$

$J^*(x_k)$ is the minimal value of cost function and is called value function.The optimal $\mathbf{u_k^*}$ control sequence is yielded

$$\mathbf{u}_k^* = \{u_k^*, u_{k+1}^*, ..., u_{k+N-1}^*\} \tag{7}$$

and only the first element in the optimal sequence is applied to the system at step k. This defines an implicit model predictive control law knows as receding horizon control strategy

$$u_k^* = \kappa(x_k) \tag{8}$$

Then closed loop system is defined

$$x_{k+1} = (A + B\kappa(x_k))x_k \tag{9}$$

2 STABILITY

The concept of stability is central in the study of dynamical systems. While stability may not necessarily guarantee optimal performance of a system, it is not conceivable that a dynamical system may perform well without being stable. MPC controllers should result in closed loops which are stable. Only stabilizing MPC controllers should be considered in the design. Stability is an overriding requirement and a lot of works has been devoted to finding sufficient conditions for stability, resulting in many of proposals for MPC.

2.1 History

Kalman observed *(1960)* that optimality does not imply stability, but under certain conditions infinite horizon optimal controllers are stabilizing. Note that the first solution that guaranteed stability consisted in using an infinite horizon cost function. Solving this open-loop optimal control problem is not usually practical, especially online because is too computationally demanding. Later research was concerned about formulation of receding MPC strategy whose solution provides stabilizing controls. *Kleinman, Thomas and Kwon et al.* are the first ones who show that addition of a terminal constraint ensures stability, but limited, because stability arguments are appropriate only for system without inputs and states constraints. Stability was a direct result of implementing stability constraint $x(T)=0$ in the optimization problem by employing the terminal condition $M(T)=0$ where M is inverse of *Riccati* variable P. The first industrial designers of MPC did not ensure stability theoretically but were aware of its importance. Before *Lyapunov* techniques were employed, researches studied the effect of control and cost horizon and cost parameters on stability when system is linear, the cost quadratic and constraints are absent. This approach was criticized in *Bitmead at al. (1990)* as a "playing games" instead of enforcing stability by modification of the optimal control problem. During *1970s* and the early *1980s* is the *Lyapunov stability theory* the most important tool neglected in the MPC with a notable acceptation. Most approaches for proving stability follow in spirit the arguments of *Keerthi* and *Gilbert (1988)* who first

employed the value function as a *Lyapunov function* for establishing stability of MPC for constrained, nonlinear and time-varying systems. Thereafter the value function was almost universally employed as a natural *Lyapunov function* for stability analysis of MPC. Several solutions for modifying the open loop optimal control problem employed in model predictive control of constrained systems so that closed loop stability could be guaranteed.

2.2 Stabilizing modifications of MPC optimization problem

In *1990s* were introduced many proposals for modifying the MPC problem so that closed loop stability for constrained/unconstrained and linear/non-linear models can be guaranteed.

The four most important modifications correspond in the main to proposals for the terminal cost F and the terminal constraints set X_f.

1. ***Terminal equality constraint***

In the version of terminal equality constraints, the terminal cost and terminal constraints satisfy $F(x_{k+N}) = 0$ (no terminal cost) and $X_f = \{0\}$. The local controller must maintain the state in X_f so $\kappa(x_k) = 0$ because zero control maintains the state at the origin.

2. ***Terminal cost function***

Another design was without terminal constraints $X_f = R^n$ and with the addition of a terminal cost function $F(x_{k+N})$. This method was first used by *Bitmead (1990)* in the context of predictive control of linear systems which was unconstrained with choice that $F(x_{k+N}) = \frac{1}{2} x_{k+N}^T P x_{k+N}$ where matrix P is solution of the Algebraic Riccati Equation. This choice yields a receding horizon controller $\kappa(x_k) = K_N x_k$ that is stabilizing. *Rawlings & Muske (1993)* made a proposal that terminal cost function $F(x_{k+N})$ is chosen to be infinite value function and $\kappa(x_k) = 0$. It guarantees stability for linear and stable systems.

3. *Terminal constraints set*

The third version is terminal constraints set. The goal of the predictive controller is to steer the last predicted state to X_f in finite time. X_f is a sub-set of R^n and contains a neighborhood of the origin. This form is sometimes referred as dual model because inside X_f is employed local stabilizing controller. It was proposed in *Michalska & Mayne (1993)* for constrained, continuous-time and non-linear systems.

4. *Terminal cost and constraints set*

The connection of terminal constraints set and terminal cost function is today the standard way of formulating the MPC controller and is denoted as *general MPC*. It has superior performance when compared to terminal constraints set MPC and can handle wide range of problems than terminal cost MPC. This proposal was extended to nonlinear and hybrid systems. The best is if $F(x_{k+N})$infinite horizon value function is. The earliest proposals are in paper of *Sznaier & Damborg 1987*. $F(x_{k+N})$ and $\kappa(x_k) = Kx_k$ are chosen to be value function and optimal controller for unconstrained infinite horizon so it is a standard LQ problem and can be solved easily. Set X_f is chosen to be maximal invariant terminal set.

2.3 Lyapunov stability

Lyapunov stability criterion is a general and useful method for characterizing and studying stability of systems.

Consider an autonomous discrete system

$$x_{k+1} = f(x_k)$$

Suppose that x_{eq} is an equilibrium point of the system so if $f(x_k) = x_{eq}$ then it remains equal to x_{eq} for all time. Our goal is to study stability of this equilibrium point. By shifting the origin of the system we assume that equilibrium point of interest occurs $x_{eq} = 0$

Definition 1: The equilibrium point $x_{eq} = 0$ of $x_{k+1} = f(x_k)$ is

1. stable if for each $\varepsilon > 0$ if there exist $\delta > 0$ such that

   $\|x_0\| < \delta \Rightarrow \|x_k\| < \varepsilon, \forall k \geq 0$

2. asymptotically stable if is stable and if $\exists \delta > 0$ such that

   $\|x_0\| < \delta \Rightarrow \lim_{k \to \infty} x_k = 0$

3. unstable if is not stable

Theorem 1*: (Lyapunov's Direct Method)*

Let $x_{eq} = 0$ be an equilibrium point for $x_{k+1} = f(x_k)$ and $V(x_k): D \to R^n$ which is continuously differenciable function such that

1. $V(0) = 0$ and $V(x) > 0 \ \forall x \in D - \{0\}$
2. $V(x_{k+1}) - V(x_k) \leq 0 \ \forall x \in D$

Then $x_{eq} = 0$ is stable and moreover

3. $V(x_{k+1}) - V(x_k) < 0 \ \forall x \in D - \{0\}$ then $x_{eq} = 0$ is asymptotically stable

3 STABILITY ANALYSIS

System (9) is not stable necessarily and feasible for any prediction horizon *N*. If the terminal weight matrix P in terminal cost function is chosen such that cost function $J(x_k, \mathbf{u_k})$ is a Lyapunov function then system is stabilizing for any horizon but solution of problem (5) may yield large inputs which infringe constraints (2). Goal is to modify optimization problem (5) such that we are able to guarantee stability for the shortest length of prediction horizon and in addition all constraints are satisfied.

The first proposal in the terminal cost framework was made by *Bitmead (1990)* who showed that choice of matrix P as solution of Algebraic *Riccati Equation (ARE)* yields a value function $J^*(x_k)$ and receding horizon controller $\kappa(x_k) = Kx_k$ such that stability is achieved. P is usually solution of the unconstrained infinite LQR problem with weights Q, R.

$$P = (A + BK)^T P(A + BK) + K^T RK + Q \tag{10}$$

One possibility is to set $K = 0$ *(Rawlings and Muske, 1993)* or $K = K_{LQ}$ where K is again calculated by infinite LQR problem *(Chmielewski and Manousiothakis, 1996; Scokaert and Rawlings, 1998)*

Consider problem (5), (6) and optimal control sequence (7) and shifted sequence of controls u_{k+1} where last element is obtained by feedback control law with $K = 0$

$$\mathbf{u}^{\mathbf{0}}_{\mathbf{k+1}} = \{u^*_{k+1}, u^*_{k+2}, \ldots\ldots, u^*_{k+N-1}, 0\} \tag{11}$$

Theorem 2. The closed-loop system (9) is asymptotically stable if

$$A^T PA + Q - P \geq 0 \tag{12}$$

and terminal states resides in the stable neighborhood of the origin (in the set of admissible states).

Proof: The demonstration results from the standard Lyapunov stability condition that value cost is decreasing function. Then closed loop system (9) is asymptotically stable if

$$J^*(x_{k+1}) - J^*(x_k) < 0, \forall x_k \in X \tag{13}$$

$$\begin{aligned} &J^*(x_{k+1}) - J^*(x_k) = J(x_{k+1}, \mathbf{u}^*_{\mathbf{k+1}}) - J(x_k, \mathbf{u}^*_{\mathbf{k}}) \le J(x_{k+1}, \mathbf{u}^0_{\mathbf{k+1}}) - J(x_k, \mathbf{u}^*_{\mathbf{k}}) = \\ &= x^T_{k+N+1} P x_{k+N+1} + \sum_{i=1}^{N} \left(x^T_{k+i} Q x_{k+i} + u^{0^T}_{k+i} R u^0_{k+i} \right) - \\ &- x^T_{k+N} P x_{k+N} - \sum_{i=0}^{N-1} \left(x^T_{k+i} Q x_{k+i} + u^{*T}_{k+i} R u^*_{k+i} \right) = \\ &= x^T_{k+N+1} P x_{k+N+1} + x^T_{k+N} Q x_{k+N} + u^{0^T}_{k+N} R u^0_{k+N} - x^T_{k+N} P x_{k+N} - x^T_k Q x_k - u^{*T}_k R u^*_k \le 0 \end{aligned} \tag{14}$$

Since $u^0_{k+N} = 0$ and $x_{k+N+1} = A x_{k+N}$ can be derived

$$- x^T_{k+N} \left(- A^T P A - Q + P \right) x_{k+N} - x^T_k Q x_k - u^{*T}_k R u^*_k \le 0 \tag{15}$$

Because of the matrices Q, R are assumed to be positive definite, the last two elements are always negative and only next condition to ensure stability has to be made

$$P - Q - A^T P A \ge 0 \tag{16}$$

If we use equality, this condition is equivalent to standard *discrete-time Lyapunov equation*. The fact that last predicted state has to be in stable neighborhood of the origin is reason why for the initial state far away from admissible set is problem (5) infeasible if predicted horizon is not long enough. To guarantee stability we have to find an admissible set and use it as terminal constraints set. It is now significant that using terminal controller $u = 0$ modify the method from using only terminal cost function to guarantee stability to problem which is combination of terminal cost function and terminal constraints set.

In the sequel a new sequence of controls that is feasible at sampling instant $k+1$ is presented which does not required residing of final state in stable neighborhood of origin. Consider system (1), (2), problem (5), (6) and optimal control sequence (7) and the following shifted sequence where $u_{k+N} = Kx_k, K \neq 0$

$$\mathbf{u_{k+1}} = \left\{u^*_{k+1}, u^*_{k+2}, \ldots\ldots, u^*_{k+N-1}, u_{k+N}\right\} \tag{17}$$

which is always feasible at step $k+1$. Let define control set U_N^k as

$$U_N^k := \left\{u_{k+N} \in U \mid J\left(x_{k+1}, \mathbf{u_{k+1}}\right) - J\left(x_k, \mathbf{u_k^*}\right) < 0\right\} \tag{18}$$

This set contains all admissible terminal controls which provide decreasing of value function.

Following theorem shows how to appropriately chose the terminal weight matrix P in order to asymptotically stabilizes system (1),(2) by the solution of problem (5), (6)

Theorem 3. The closed loop system (9) with the prediction horizon N at each sampling time is globally asymptotically stable if

$$A^T PB\left(R + B^T PB\right)^{-1} B^T PA - A^T PA - Q + P \geq 0 \tag{19}$$

$$U_N^k \neq \emptyset, \quad \forall k \tag{20}$$

Proof: The aim is to prove that conditions (19), (20) are sufficient for ensuring decreasing of value function what is required for guarantying asymptotic stability and results from Lyapunov stability theory.

Suppose that $J^*(x_k)$ is value function in step k and $J^*(x_{k+1})$ in next step $k+1$, then if closed loop system (9) is asymptotically stable next condition is valid.

$$J^*(x_{k+1}) - J^*(x_k) = J\left(x_{k+1}, \mathbf{u^*_{k+1}}\right) - J\left(x_k, \mathbf{u_k^*}\right) < 0, \forall x \in X \tag{21}$$

The goal is to show that this inequality (21) is suitable for every $u_{k+N} \in U_N^k$ and then we don't need any others condition for choosing terminal control, just to be included in U_N^k.

$$J\left(x_{k+1}, \mathbf{u}_{\mathbf{k+1}}^*\right) - J\left(x_k, \mathbf{u}_{\mathbf{k}}^*\right) \le J\left(x_{k+1}, \mathbf{u}_{k+1}\right) - J\left(x_k, \mathbf{u}_{\mathbf{k}}^*\right) < 0 \tag{22}$$

$$\begin{aligned} &J\left(x_{k+1}, \mathbf{u}_{k+1}\right) - J\left(x_k, \mathbf{u}_{\mathbf{k}}^*\right) = \\ &= x_{k+N+1}^T P x_{k+N+1} + \sum_{i=1}^{N}\left(x_{k+i}^T Q x_{k+i} + u_{k+i}^T R u_{k+i}\right) - x_{k+N}^T P x_{k+N} - \sum_{i=0}^{N-1}\left(x_{k+i}^T Q x_{k+i} + u_{k+i}^{*T} R u_{k+i}^*\right) = \\ &= x_{k+N+1}^T P x_{k+N+1} + x_{k+N}^T Q x_{k+N} + u_{k+N}^T R u_{k+N} - x_{k+N}^T P x_{k+N} - x_k^T Q x_k - u_k^{*T} R u_k^* < 0 \end{aligned} \tag{23}$$

Last two elements are always negative because matrices Q, R are assumed to be positive definite so the core of proof is to derive the condition for the first four elements.

$$\begin{aligned} &x_{k+N+1}^T P x_{k+N+1} + x_{k+N}^T Q x_{k+N} + u_{k+N}^T R u_{k+N} - x_{k+N}^T P x_{k+N} = \\ &= \left(A x_{k+N} + B u_{k+N}\right)^T P \left(A x_{k+N} + B u_{k+N}\right) + x_{k+N}^T Q x_{k+N} + u_{k+N}^T R u_{k+N} - x_{k+N}^T P x_{k+N} = \\ &= u_{k+N}^T \left(R + B^T P B\right) u_{k+N} + 2 x_{k+N}^T A^T P B u_{k+N} + x_{k+N}^T \left(A^T P A + Q - P\right) x_{k+N} = f\left(u_{k+N}\right) \le 0 \end{aligned} \tag{24}$$

The optimal final control u_{k+N}^* is given by

$$\begin{aligned} &\frac{\partial f\left(u_{k+N}\right)}{\partial u_{k+N}} = 2 u_{k+N}\left(R + B^T P B\right) + 2 B^T P A x_{k+N} \Leftrightarrow \tilde{u}_{k+N} = -\left(R + B^T P B\right)^{-1} B^T P A x_{k+N} \\ &= K_{LQ} x_{k+N} \end{aligned} \tag{25}$$

Optimal control is defined by LQR feedback and corresponding minimal value of function $f\left(u_{k+N}\right)$ is

$$f\left(\tilde{u}_{k+N}\right) = -x_{k+N}^T \left(A^T P B \left(R + B^T P B\right)^{-1} B^T P A - A^T P A - Q + P\right) x_{k+N} \le 0 \tag{26}$$

and this is equivalent to condition (19).

If inequality is solved as equality, standard *Discrete Algebraic Riccati Equation (DARE)* is obtained and problem is equal to infinite horizon LQR problem.

Most important fact is that presented results are sufficient for stability unconstrained system. For providing stability for system (1) with constraints (2) is existence of non empty set U_N^k necessary, so at least one u_{k+N} satisfy condition (22) and at the same time input constraints. Advantage against method which guarantees stability with terminal constraints set is no increasing the difficulty of the optimization problem (5) (6) by adding constraints for terminal states and no problems with constructing this terminal set. Presented method also provides shorter stabilizing prediction horizon since the set of admissible and stabilizing terminal controls depends on the final state and the range is changing at each sampling time.

4 STABILIZING CONDITIONS

4.1 Single input systems

If system is single input linear system then the set of stabilizing and feasible controls (18) can be calculated straightforwardly just by using basic mathematical actions. Functions (24) becomes a scalar quadratic function and under the condition that first element is positive then all inputs which guarantee stability are contained between the roots of this quadratic function. Existence of inputs limits causes infeasibility of some or all of these inputs for prediction horizon *N*. To obtain the set of stabilizing and feasible controls U_N^k is necessary to do intersection between roots interval and the set of admissible inputs U.

$$R + B^T PB > 0 \tag{27}$$

$$U_N^k = \langle \min(u_1, u_2), \max(u_1, u_2) \rangle \cap U \tag{28}$$

where u_1, u_2 are roots of function (24)

$$D = x_{x+N}^T \left(A^T PBB^T PA - \left(R + B^T PB \right) \left(A^T PA + Q - P \right) \right) x_{k+N} \tag{29}$$

$$\begin{aligned} u_1 &= \left(R + B^T PB \right)^{-1} \left(- x_{k+N}^T A^T PB - \sqrt{D} \right) \\ u_2 &= \left(R + B^T PB \right)^{-1} \left(- x_{k+N}^T A^T PB + \sqrt{D} \right) \end{aligned} \tag{30}$$

Set U_N^k is excellent tool how to investigate the shortest stabilizing and feasible horizon. Just for given initial state find the first horizon for which is the set of controls U_N^k nonempty.

If P is chosen as solution of Discrete Algebraic Riccati Equation, it means that inequality (19) is calculated as equality, then discriminant is equal to zero and function (24) has only one root $u_1 = u_2$. After that we have to find out if this root satisfies input

constraints. If yes, then predictive controller is stabilizing and feasible for given prediction horizon.

This alternative of choosing P is not appropriate, because existence of only one stabilizing and feasible terminal input is limiting. Therefore is better to solve inequality (19) as *Linear Matrix Inequality (LMI).* By means of this method not only matrix P is calculated but also a state-feedback matrix K, which is used later for local controller to demonstrate the advantages of this stabilizing approach comparing to terminal set.

Inequality (19) is equal to following expressions

$$K = -(R + B^T PB)^{-1} B^T PA \tag{31}$$

$$P - \left(A + BK\right)^T P(A + BK) - K^T RK - Q \geq 0 \tag{32}$$

This problem can be rewritten as an semi-definite program by using *Schur complements* and new variables Z, Y that are related according to $Z = P^{-1}, Y = KP^{-1}$.

If Z, Y are positive definite then (31), (32) are equivalent to:

$$\begin{pmatrix} Z & Z & Y^T & (AZ + BY)^T \\ Z & Q^{-1} & 0 & 0 \\ Y & 0 & R^{-1} & 0 \\ (AZ + BY) & 0 & 0 & Z \end{pmatrix} \geq 0 \tag{33}$$

Proof is based on applying the Schur complement to (33) which yields:

$$\begin{aligned} Z - \begin{pmatrix} Z & Y^T & (AZ + BY)^T \end{pmatrix} \begin{pmatrix} Q & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & Z^{-1} \end{pmatrix} \begin{pmatrix} Z \\ Y \\ (AZ + BY) \end{pmatrix} = \\ = Z - ZQZ - Y^T RY - (AZ + BY)^T Z^{-1} (AZ + BY) \end{aligned} \tag{34}$$

Substituting Z and Y and pre-multiplying and post-multiplying with P^{-1} yields the equivalent matrix inequality (32).

4.1.1 Infinity norm

In MPC theory quadratic cost function and 2-norms are not implemented all the time. Sometimes it is more efficient to work with 1 and infinity norms because the optimization problem can then be solved by linear programming. Utilization is also in hybrid systems where reduction of the on-line computational complexity comparing to 2-norms is significant.

Some assumptions are needed to derive very simple formulation of stabilizing and feasible terminal inputs for linear single input system as in previous section. In this thesis is not shown how to calculate terminal penalty matrix P in case of MPC based on infinity norms is used. For details see *Christophersen & Morari (2007) and Lazar et al. (2005).* Suppose that matrix P guarantee stability in terms of decreasing value function. Another assumption is that matrix R is equal to 0. In next steps the interval of stabilizing controls is deduced and set U_N^k is defined. Inequality (24) can be rewritten into following expression

$$\left\|P\left(Ax_{k+N}+Bu_{k+N}\right)\right\|_\infty+\left\|Qx_{k+N}\right\|_\infty-\left\|Px_{k+N}\right\|_\infty\leq 0 \tag{35}$$

Last two elements are scalars and do not contain unknown variable u_{k+N}, so very simple inequality is obtained. Unknown variable u_{k+N} is inside the infinity norm of vector $P\left(Ax_{k+N}+Bu_{k+N}\right)$ and is function of terminal state x_{k+N}. Infinity norm is choosing the maximal absolute element of mentioned vector. We cannot guarantee which element of this vector is maximal in advance, therefore all possibilities are assumed. Then infinity norm can be eliminated and result is system of very simple linear inequalities for each row of vector $P\left(Ax_{k+N}+Bu_{k+N}\right)$.

$$
\begin{aligned}
&+\left[P\left(Ax_{k+N}+Bu_{k+N}\right)\right]_1 \le \left\|Px_{k+N}\right\|_\infty - \left\|Qx_{k+N}\right\|_\infty \\
&-\left[P\left(Ax_{k+N}+Bu_{k+N}\right)\right]_1 \le \left\|Px_{k+N}\right\|_\infty - \left\|Qx_{k+N}\right\|_\infty \\
&+\left[P\left(Ax_{k+N}+Bu_{k+N}\right)\right]_2 \le \left\|Px_{k+N}\right\|_\infty - \left\|Qx_{k+N}\right\|_\infty \\
&-\left[P\left(Ax_{k+N}+Bu_{k+N}\right)\right]_2 \le \left\|Px_{k+N}\right\|_\infty - \left\|Qx_{k+N}\right\|_\infty \\
&\vdots \\
&+\left[P\left(Ax_{k+N}+Bu_{k+N}\right)\right]_n \le \left\|Px_{k+N}\right\|_\infty - \left\|Qx_{k+N}\right\|_\infty \\
&-\left[P\left(Ax_{k+N}+Bu_{k+N}\right)\right]_n \le \left\|Px_{k+N}\right\|_\infty - \left\|Qx_{k+N}\right\|_\infty
\end{aligned}
\tag{36}
$$

where $[\cdot]_i$ denotes the i-th row of a vector or a matrix and n is order of system.

From inequalities for each row some sub-interval for u_{k+N} is calculated.

In terms of transparency let's make a substitution that $z = \left\|Px_{k+N}\right\|_\infty - \left\|Qx_{k+N}\right\|_\infty$

$$
\begin{aligned}
&\left\langle u_{k+N}\right\rangle_1 \in \left\langle (PB)_1^{-1}\left(z-\left(PAx_{k+N}\right)_1\right) \quad ,(PB)_1^{-1}\left(-z-\left(PAx_{k+N}\right)_1\right)\right\rangle \\
&\left\langle u_{k+N}\right\rangle_2 \in \left\langle (PB)_2^{-1}\left(z-\left(PAx_{k+N}\right)_2\right) \quad ,(PB)_2^{-1}\left(-z-\left(PAx_{k+N}\right)_2\right)\right\rangle \\
&\vdots \\
&\left\langle u_{k+N}\right\rangle_n \in \left\langle (PB)_n^{-1}\left(z-\left(PAx_{k+N}\right)_n\right) \quad ,(PB)_n^{-1}\left(-z-\left(PAx_{k+N}\right)_n\right)\right\rangle
\end{aligned}
\tag{37}
$$

The final interval of stabilizing controls, independent on result of infinity norm of vector $\left\|P\left(Ax_{k+N}+Bu_{k+N}\right)\right\|_\infty$, is calculated as intersection of sub-intervals $\left(u_{k+N}\right)_i, i = 1...n$

$$
\left\langle u_{k+N}\right\rangle = \left(u_{k+N}\right)_1 \cap \left(u_{k+N}\right)_2 \cdots \cap \left(u_{k+N}\right)_n \tag{38}
$$

The final step is comparing this interval to input constraints set U .

$$
U_N^k = \left\langle u_{k+N}\right\rangle \cap U \tag{39}
$$

4.2 Multi input system

It is much more difficult to obtain the set of controls U_N^k in case multi input system because equation (24) is no longer a simply quadratic function but a function for volume of ellipse if we have 2 inputs or of ellipsoid in case with more inputs. Constructing stabilizing intervals is not easy and finding their intersection with the set of admissible inputs is also not an ordinary business.

Simplified solution how to acquire stabilizing terminal inputs is proposed. Strategy is based on constructing polytopic sub-set of stabilizing inputs. In case when number of inputs is same as number of states is possible to calculate an independent stabilizing sub-interval for each input. The idea is developed in next steps. Fundamental is the existence of Cholesky factorization of matrices P and $P-Q$. Consider the predictive cost function (4) with the matrix $R=0$ which is not so unusual choice if we are interested in a bounded u_k.

Then inequality (24) can be rewritten

$$\begin{aligned}
&\left(Ax_{k+N}+Bu_{k+N}\right)^T P\left(Ax_{k+N}+Bu_{k+N}\right)-x_{k+N}^T\left(P-Q\right)x_{k+N}= \\
&=\left[\tilde{P}\left(Ax_{k+N}+Bu_{k+N}\right)\right]^T\left[\tilde{P}\left(Ax_{k+N}+Bu_{k+N}\right)\right]-\left[P_Q\tilde{x}_{k+N}\right]^T\left[P_Q\tilde{x}_{k+N}\right]= \\
&=\left(\left[\tilde{P}\left(Ax_{k+N}+Bu_{k+N}\right)\right]+\left[P_Q\tilde{x}_{k+N}\right]\right)^T\left(\left[\tilde{P}\left(Ax_{k+N}+Bu_{k+N}\right)\right]-\left[P_Q\tilde{x}_{k+N}\right]\right)\le 0
\end{aligned} \tag{40}$$

where matrices $\tilde{P},\tilde{P}_Q$ are result of Cholesky factorization:

$$\tilde{P}=chol(P)$$
$$\tilde{P}_Q=chol(P-Q)$$

Both matrices are upper triangular with dimension equal to number of states n and is assumed that $P-Q$ is positive definite so equation (40) is converted to

$$
\begin{aligned}
&\left[\left(\tilde{P}Ax_{k+N}\right)_1+\left(\tilde{P}Bu_{k+N}\right)_1+\left(\tilde{P}_Q x_{k+N}\right)_1\right]\left[\left(\tilde{P}Ax_{k+N}\right)_1+\left(\tilde{P}Bu_{k+N}\right)_1-\left(\tilde{P}_Q x_{k+N}\right)_1\right]+\\
&+\left[\left(\tilde{P}Ax_{k+N}\right)_2+\left(\tilde{P}Bu_{k+N}\right)_2+\left(\tilde{P}_Q x_{k+N}\right)_2\right]\left[\left(\tilde{P}Ax_{k+N}\right)_2+\left(\tilde{P}Bu_{k+N}\right)_2-\left(\tilde{P}_Q x_{k+N}\right)_2\right]+\\
&\vdots\\
&+\left[\left(\tilde{P}Ax_{k+N}\right)_n+\left(\tilde{P}Bu_{k+N}\right)_n+\left(\tilde{P}_Q x_{k+N}\right)_n\right]\left[\left(\tilde{P}Ax_{k+N}\right)_n+\left(\tilde{P}Bu_{k+N}\right)_n-\left(\tilde{P}_Q x_{k+N}\right)_n\right]\le 0
\end{aligned}
\tag{41}
$$

where $[\cdot]_i$ denotes the i-th row of that matrix.

The goal of this method is to divide inequality (41) into collection of more easily inequalities.

$$
\begin{aligned}
&\left[\left(\tilde{P}Ax_{k+N}\right)_i+\left(\tilde{P}Bu_{k+N}\right)_i+\left(\tilde{P}_Q x_{k+N}\right)_i\right]\left[\left(\tilde{P}Ax_{k+N}\right)_i+\left(\tilde{P}Bu_{k+N}\right)_i-\left(\tilde{P}_Q x_{k+N}\right)_i\right]=\\
&=\left[\left(\left(\tilde{P}A+\tilde{P}_Q\right)x_{k+N}\right)_i+\left(\tilde{P}Bu_{k+N}\right)_i\right]\left[\left(\left(\tilde{P}A-\tilde{P}_Q\right)x_{k+N}\right)_i+\left(\tilde{P}Bu_{k+N}\right)_i\right]\le 0
\end{aligned}
\tag{42}
$$

$i=1\ldots n$

In the next step the solution of each inequality (42) is found

$$
\begin{aligned}
&u_1^i\le\left(\tilde{P}Bu_{k+N}\right)_i\le u_2^i\\
&u_1^i=\min\left[-\left(\left(\tilde{P}A+\tilde{P}_Q\right)x_{k+N}\right)_i,-\left(\left(\tilde{P}A-\tilde{P}_Q\right)x_{k+N}\right)_i\right]\\
&u_2^i=\max\left[-\left(\left(\tilde{P}A+\tilde{P}_Q\right)x_{k+N}\right)_i,-\left(\left(\tilde{P}A-\tilde{P}_Q\right)x_{k+N}\right)_i\right]
\end{aligned}
\tag{43}
$$

Let's denote $d_{i,j}$ each element of matrix $\tilde{P}B\subset R^{n\times m}$

$$\tilde{P}B = \begin{bmatrix} d_{1,1} & d_{1,2} & \cdots & d_{1,m} \\ d_{2,1} & d_{2,2} & \cdots & d_{2,m} \\ \vdots & & & \\ d_{n,1} & d_{n,2} & & d_{n,m} \end{bmatrix} \tag{44}$$

Then the multiplication of $\tilde{P}Bu_{k+N}$ is

$$\tilde{P}Bu_{k+N} = \begin{bmatrix} d_{1,1}u_{k+N}^{1} + d_{1,2}u_{k+N}^{2} + \cdots d_{1,m}u_{k+N}^{m} \\ d_{2,1}u_{k+N}^{1} + d_{2,2}u_{k+N}^{2} + \cdots d_{2,m}u_{k+N}^{m} \\ \vdots \\ d_{n,1}u_{k+N}^{1} + d_{n,2}u_{k+N}^{2} + \cdots d_{n,m}u_{k+N}^{m} \end{bmatrix} \tag{45}$$

Now the solution (43) of equation (42) is written in complex way and can be transformed to the expression of polytope in H representation.

$$\begin{aligned} &\begin{bmatrix} u_1^1 \\ u_1^2 \\ \vdots \\ u_1^n \end{bmatrix} \le \tilde{P}B \begin{bmatrix} u_{k+N}^1 \\ u_{k+N}^2 \\ \vdots \\ u_{k+N}^m \end{bmatrix} \le \begin{bmatrix} u_2^1 \\ d_2^2 \\ \vdots \\ d_2^n \end{bmatrix} \Leftrightarrow \\ &\Leftrightarrow \begin{bmatrix} u_1^1 \\ u_1^2 \\ \vdots \\ u_1^n \end{bmatrix} \le \begin{bmatrix} d_{1,1}u_{k+N}^{1} + d_{1,2}u_{k+N}^{2} + \cdots d_{1,m}u_{k+N}^{m} \\ d_{2,1}u_{k+N}^{1} + d_{2,2}u_{k+N}^{2} + \cdots d_{2,m}u_{k+N}^{m} \\ \vdots \\ d_{n,1}u_{k+N}^{1} + d_{n,2}u_{k+N}^{2} + \cdots d_{n,m}u_{k+N}^{m} \end{bmatrix} \le \begin{bmatrix} u_2^1 \\ u_2^2 \\ \vdots \\ u_2^n \end{bmatrix} \Leftrightarrow \\ &\Leftrightarrow U_1 \le \tilde{P}Bu_{k+N} \le U_2 \end{aligned} \tag{46}$$

Polytope P represents the sub-set of stabilizing terminal controls and can be established as

$$\mathrm{P} = \left\{ u_{k+N} \in R^m \middle| \begin{bmatrix} \tilde{P}B \\ -\tilde{P}B \end{bmatrix} u_{k+N} \le \begin{bmatrix} U_2 \\ -U_1 \end{bmatrix} \right\} \tag{47}$$

Admissible inputs are given by constraint set U, which is usually for MIMO systems defined again as polytopic set in H representation.

Now the sub-set of set (18) which contains all stabilizing and feasible terminal inputs is defined:

$$\hat{U}_N^k := P \cap U \tag{48}$$

If system has the same number of states and inputs, it means that $n = m$ then from (43) is the interval for every input calculated very easily

$$\begin{aligned} & u_1^i \le \left(u_{k+N}\right)_i \le u_2^i \\ & u_1^i = \min\left[-\left(\left(\tilde{P}B\right)^{-1}\left(\tilde{P}A+\tilde{P}_Q\right)x_{k+N}\right)_i, -\left(\left(\tilde{P}B\right)^{-1}\left(\tilde{P}A-\tilde{P}_Q\right)x_{k+N}\right)_i\right] \\ & u_2^i = \max\left[-\left(\left(\tilde{P}B\right)^{-1}\left(\tilde{P}A+\tilde{P}_Q\right)x_{k+N}\right)_i, -\left(\left(\tilde{P}B\right)^{-1}\left(\tilde{P}A-\tilde{P}_Q\right)x_{k+N}\right)_i\right] \end{aligned} \tag{49}$$

Every stabilizing sub-interval for each input has to have intersection with particular input constraint interval and then the piece-wise linear sub-set of (18) is defined.

$$\hat{U}_N^k := \left\{u_{k+N} \in R^n \middle| u_1^i \le \left(u_{k+N}\right)_i \le u_2^i, i = 1...n\right\} \cap U \tag{50}$$

The first prediction horizon N which guarantees non-empty $\hat{U}_N^k$ is the shortest stabilizing horizon for given constrained MIMO system. It is important to mention that if we break down polytopic sub-set of stabilizing terminal controls to sub-intervals for every input then we lose some stabilizing terminal inputs. If sub-set is consequently constructed from these intervals then is smaller than sub-set obtained by constructing polytopic sub-set according to (47) because sub-set from intervals is orthogonal polytopic sub-set.

5 NUMERICAL EXAMPLES

The following theoretical and real examples from literature will be used to demonstrate the stability approach described previously and to make comparison with method where terminal constraints are implemented to guarantee stability.

5.1 SISO systems

5.1.1 Example 1 (Bemporad et al., 2000)

Consider the following second order system

$$Y(s) = \frac{2}{s^2 + 3s + 2} U(s) \tag{51}$$

Sample the dynamics with T=0.1s and obtain the state-space representation

$$x_{k+1} = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} x_k + \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix} u_k \tag{52}$$

The input and the states are subject to the following constraints

$$\begin{aligned} &-2 \le u_k \le 2 \\ &x_k \ge [-0.5 \; -0.5]^T \end{aligned} \tag{53}$$

Consider the predictive cost (4) with $Q = I, R = 0.01$ and initial state $x_0 = [1 \quad 1]^T$

Matrix P is calculated by using LMI (33) to satisfy the inequality (19).

$$P = \begin{bmatrix} 2.5745 & 2.0029 \\ 2.0029 & 9.0417 \end{bmatrix} \tag{54}$$

The set (interval) of stabilizing controls and the input constraints are plotted in Fig.1 as a function of prediction horizon N. The shortest horizon sufficient to guarantee stability and feasibility for given constrained system (52) is $N = 3$ because is the first one for which is intersection between displayed sets (i.e. U_N^k set) and input constraint set U non-empty.

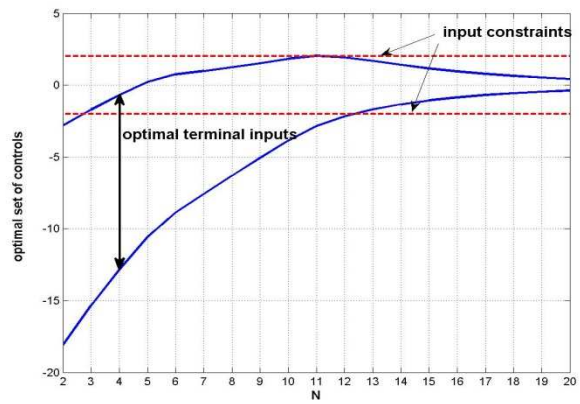

Fig.1 Optimal set of controls as function of prediction horizon for plant (52).

The performance of MPC controller for example 1 with $N = 3$ is shown in Fig. 2.

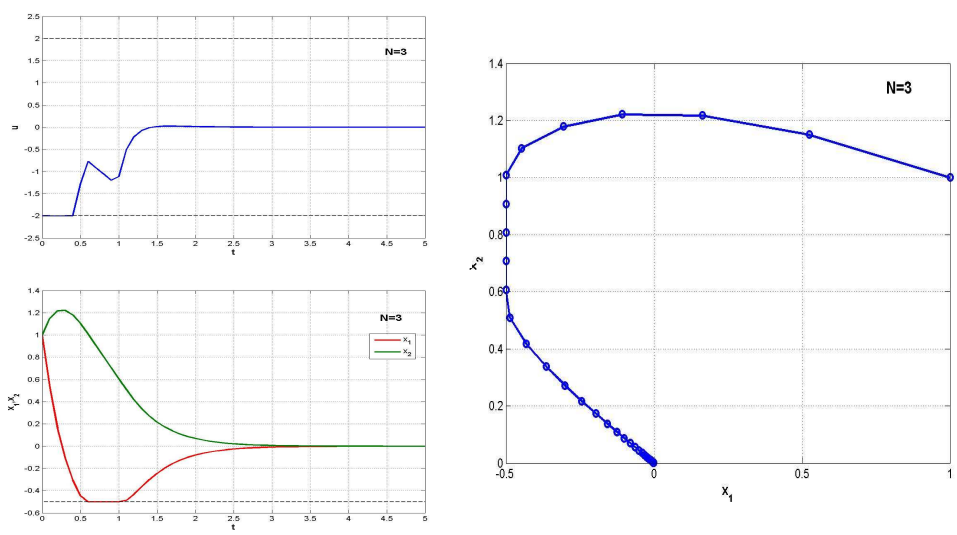

Fig. 2 State and input histories for plant (52) in closed loop with the introduced predictive control algorithm.

If terminal invariant set is implemented to MPC the sufficient length of horizon to guarantee stability and feasibility shifts from 3 to 10. This set is calculated in standard way as polytopic set by using local controller $u = Kx$, where K is result of solution LMI (33). In this example

$$K = [-7.3167 \quad -7.6992]. \tag{55}$$

Fig. 3 illustrates and compares the performance of example 1 by using different method to guarantee stability. Even the complete removal of terminal set in our approach results in almost the same trajectory of states and inputs as in concept of method with terminal constraints.

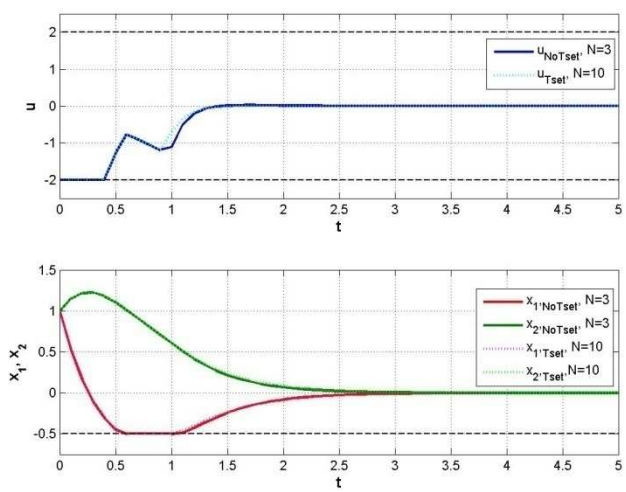

Fig. 3 Comparison performance of states and inputs for plant (52) with and without terminal set in MPC algorithm.

The reduction of prediction horizon is significant and leads to less computation rate. The other big advantage is that terminal set does not have to be constructed.

5.1.2 Example 2 (Mayne et al., 2000)

Consider next the following system, which is the zero order hold discretization with a sampling period $T_s = 0.4s$ of the double integrator

$$x_{k+1} = \begin{bmatrix} 1 & 0 \\ 0.4 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.4 \\ 0.08 \end{bmatrix} u_k \tag{56}$$

With the control input subject to constraint

$$-1 \le u_k \le 1 \tag{57}$$

The MPC tuning parameters are $Q = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}$, $R = 0.4$. Initial state is $x_0 = [2 \quad 3]^T$.

Terminal matrix P is gained again as solution of LMI (33)

$$P = \begin{bmatrix} 2.0774 & 1.1768 \\ 1.1768 & 2.6160 \end{bmatrix} \tag{58}$$

Evolution of set of stabilizing controls is shown in Fig. 4

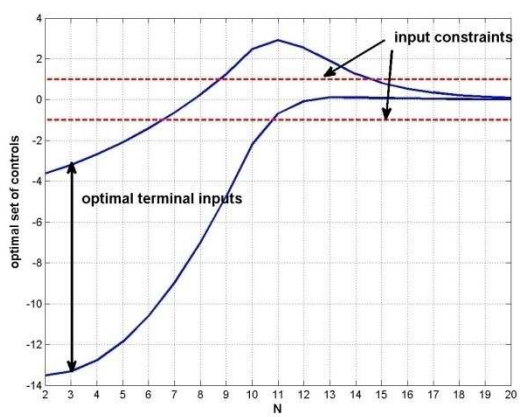

Fig.4 Optimal set of controls as function of prediction horizon for plant (56).

The shortest stabilizing and feasible horizon is $N = 7$. In Fig. 5 performance of MPC controller with this length of prediction horizon is depicted.

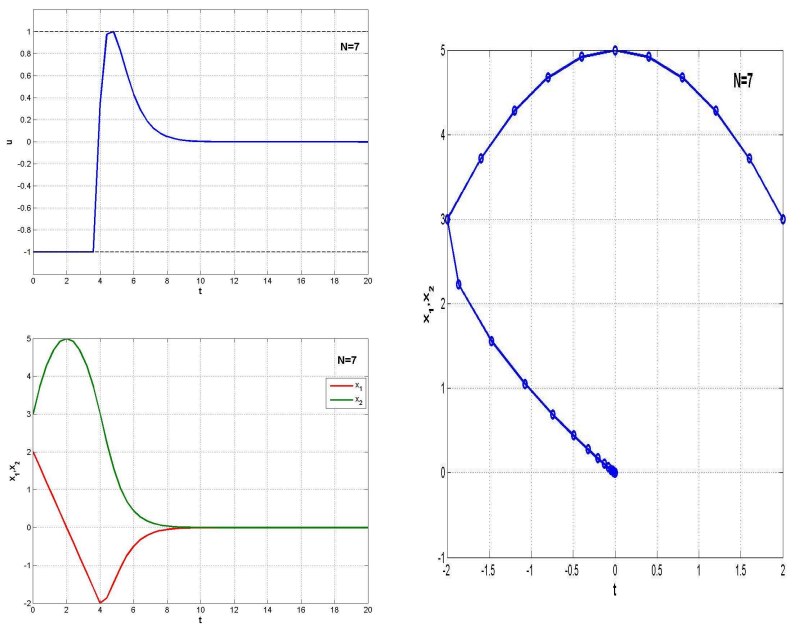

Fig. 5 State and input histories for plant (56) in closed loop with the introduced predictive control algorithm.

Again the simulations with terminal set were done and in contrast to our method, MPC algorithm with terminal constraints requires prediction horizon $N = 10$ to guarantee stability and feasibility.

$$K = \begin{bmatrix} -1.8690 & -1.1054 \end{bmatrix} \tag{59}$$

Comparison graphs are proof that states and inputs trajectories of both MPC controllers are almost same (see Fig. 6).

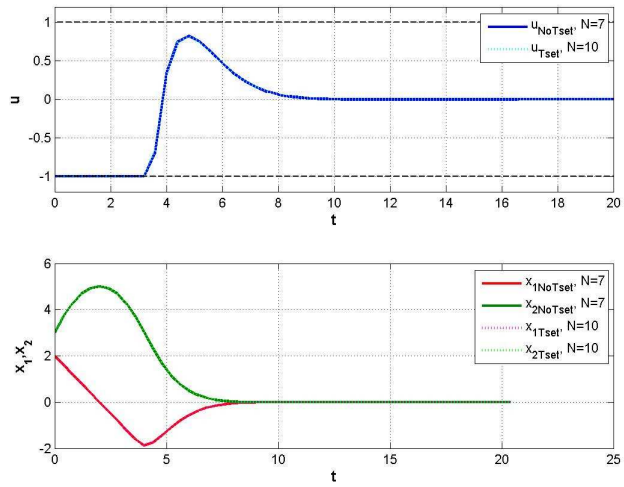

Fig. 6 Comparison performance of states and inputs for plant (56) with and without terminal set in MPC algorithm.

5.1.3 Example 3: Laboratory liquid tanks system (Rauova et al., 2009)

In this example, approach outlined previously, is applied to real physical process of liquid tanks.

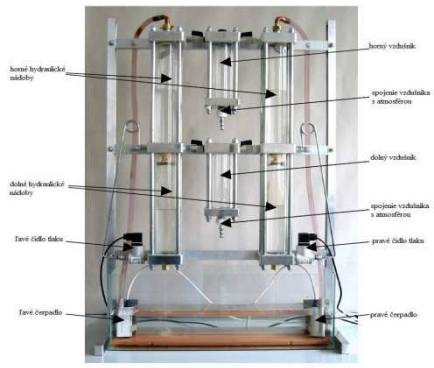

Fig.7 Laboratory liquid tanks system

The laboratory liquid tanks device as depicted in Fig. 7 is comprised of two pairs of interconnected tanks. Each pair consists of two glass tanks connected by small opening, which allows the liquid from upper tank to flow into the lower tank. Only left pair of the tanks is considered.

The mathematical model can be captured by two differential equations:

$$F\frac{dh_1}{dt} = q - k\sqrt{h_1}$$
$$F\frac{dh_2}{dt} = k\sqrt{h_1} - k\sqrt{h_2} \tag{60}$$

Where $F = 12.566cm^2$ is the cross-sectional area of the tanks, $k = 3.667cm^{2.5}s^{-1}$ denotes a constant representing the resistance of the opining connecting the tanks, h_1 a h_2 represent the liquid levels in upper and lower tank and q is liquid flow. By linearization around steady-states values $h_1^s = h_2^s = 21.44cm$, $q_s = 16.9811cm^3s^{-1}$ and by discretization with the sampling time $T_s = 5$ s the following model can be derived

$$h_{k+1} = \begin{bmatrix} 0.8543 & 0 \\ 0.1345 & 0.8543 \end{bmatrix} h_k + \begin{bmatrix} 0.3557 \\ 0.0273 \end{bmatrix} q_k \tag{61}$$

Subject to the input and states constraints

$$0 \le q_k \le 20cm^3s^{-1} \tag{62}$$
$$\begin{bmatrix} 0 & 0 \end{bmatrix}^T \le h_k \le \begin{bmatrix} 25 & 25 \end{bmatrix}^T cm$$

Penalty matrices and initial states are chosen as $Q = I$, $R = 1$ $x_0 = [20 \quad 18]^T cm$.
Following the procedure presented in theoretical part matrix P is calculated.

$$P = \begin{bmatrix} 3.7045 & 1.1001 \\ 1.1001 & 4.7447 \end{bmatrix} \tag{63}$$

Interval $(\min(u_1, u_2), \max(u_1, u_2))$ (28) for each prediction horizon is plotted in Fig. 8

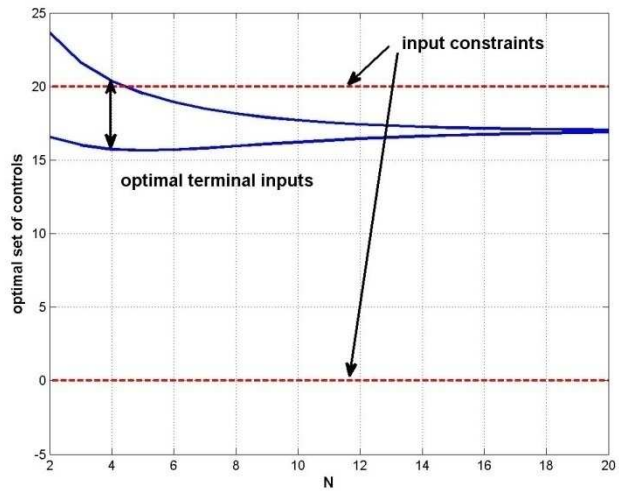

Fig.8 Optimal set of controls as function of prediction horizon for plant (61).

Notice that prediction horizon $N = 2$ is sufficient for guarantying stability of system (61). Fig. 9 provide the simulations results obtained by running the MPC controller.

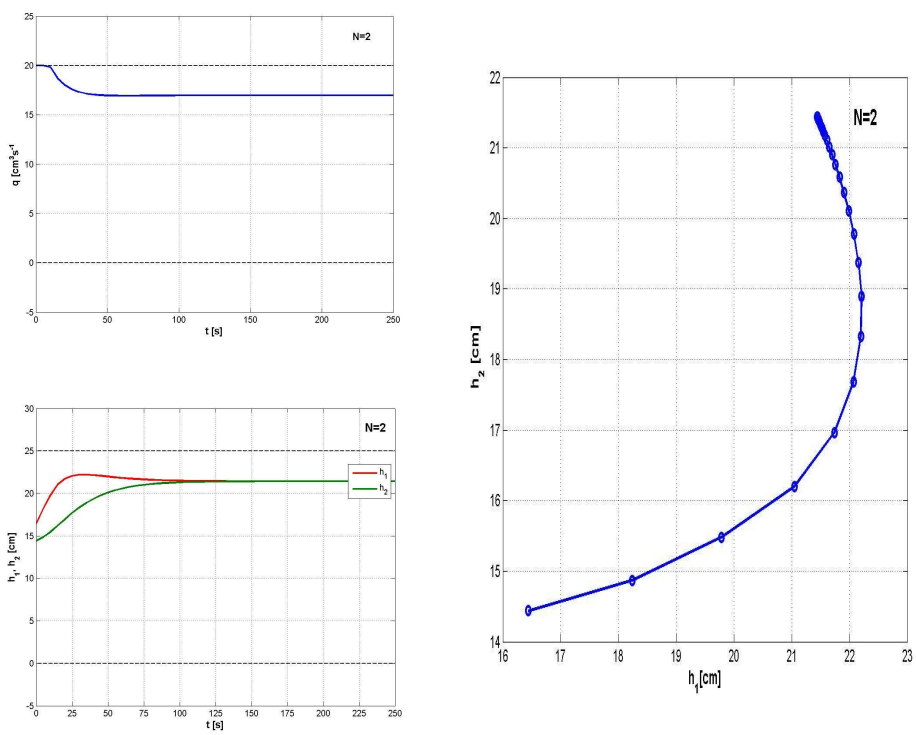

Fig. 9 State and input histories for plant (61) in closed loop with the introduced predictive control algorithm.

In the case when stability is guaranteed by terminal set the shortest prediction horizon is $N = 3$. Over again only the small differences in performance of states and inputs can be observed in comparison Fig. 10 between using terminal set and no using this set. For constructing terminal set was used this local feedback matrix

$$K = \begin{bmatrix} -0.8236 & -0.2898 \end{bmatrix} \tag{64}$$

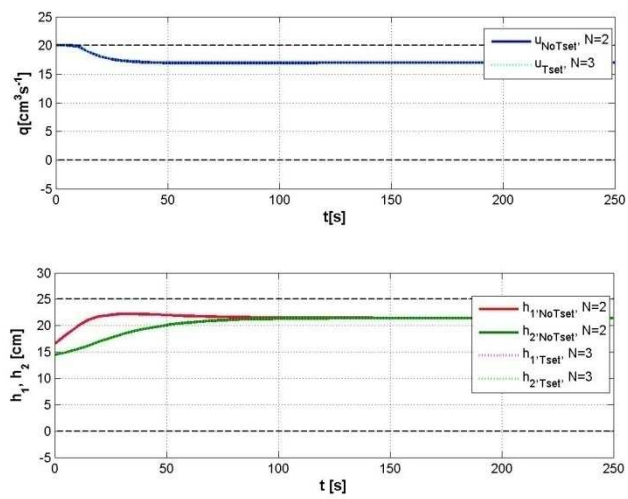

Fig. 10 Comparison performance of states and inputs for plant (61) with and without terminal set in MPC algorithm.

5.2 MIMO system

5.2.1 Example 1

Consider the double integrator with two inputs.

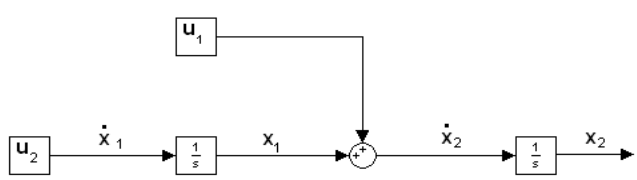

Fig. 11 Scheme of double integrator

where the sampling interval is $T_s = 0.2$ and the following state-space representation is acquired

$$x_{k+1} = \begin{bmatrix} 1 & 0 \\ 0.2 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.2 & 0 \\ 0.02 & 0.2 \end{bmatrix} u_k \tag{65}$$

States are unconstrained and inputs are subjects to the constraint

$$\begin{bmatrix} -0.5 & -0.5 \end{bmatrix}^T \le u_k \le \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}^T \tag{66}$$

Predictive cost (4) is designed with $Q = I$ and $R = 0$ and $x_0 = \begin{bmatrix} 0.8 & -0.5 \end{bmatrix}^T$

By LMI (33) matrix P is calculated. In this matrix inequality put $R = I$ although in simulation is assumed that $R = 0$. It only effects that cost function will decrease more rapidly.

$$P = \begin{bmatrix} 8.6530 & 2.5475 \\ 2.5475 & 6.9554 \end{bmatrix} \tag{67}$$

Due to the fact that number of inputs is more than one, stabilizing sub-set for each prediction horizon can be obtained. The shortest stabilizing horizon for this MIMO system (65) is deducted from Fig. 12 and is equal to 2. This is the horizon where intersection between input constraints set and stabilizing sub-set of terminal controls is non empty. On the left side are displayed polytopic sets for only first 8 horizons, because with the longer prediction horizon these sets are very small sets around zero.

On the right part are shown intersections between input constraints and polytopes of stabilizing inputs so are shown sets U_N^k.

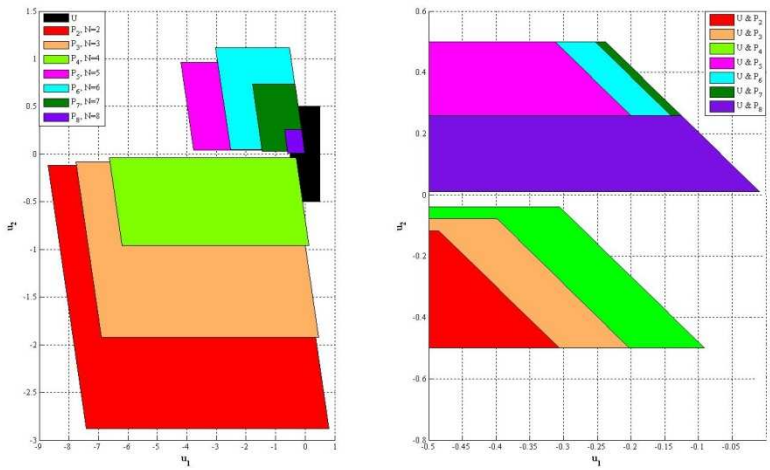

Fig.12 Polytopic stabilizing sub-sets as function of prediction horizon and intersection between input constraints set U and stabilizing sub-sets.

The simulations of states and inputs are depicted in Fig. 13.

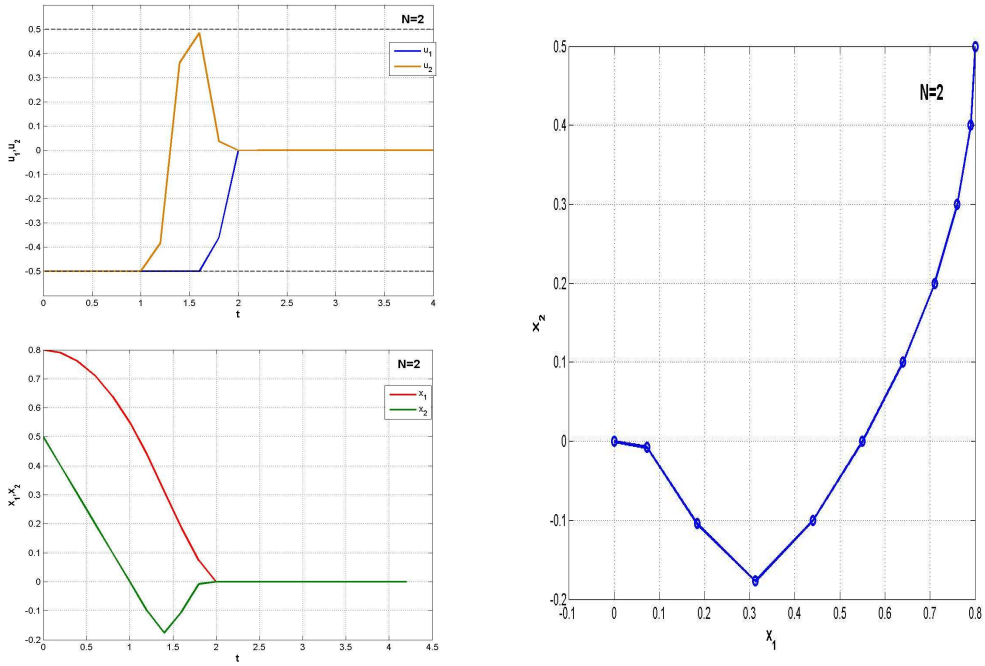

Fig. 13 State and input histories for plant (65) in closed loop with the introduced predictive control algorithm.

Using terminal set in this example leads to the increasing of the length of prediction horizon from 2 to 4 and performances of states and inputs are similar to simulations without terminal sets. (Fig. 14).

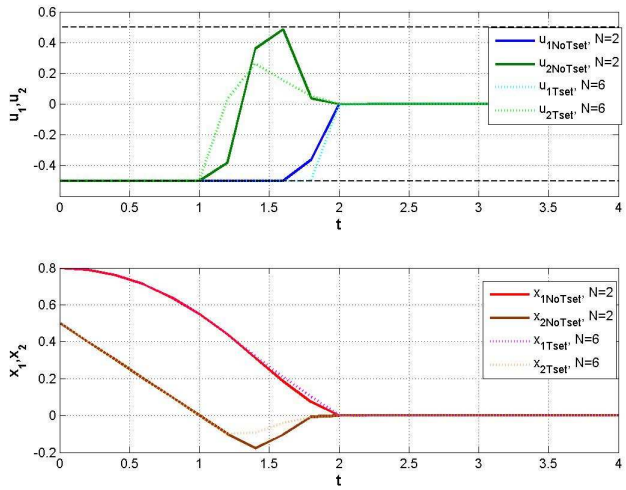

Fig. 14 Comparison performance of states and inputs for plant (65) with and without terminal set in MPC algorithm.

5.2.2 Example 2: Main fractionator of a fluid catalyc cracer unit (FCCU) (Vada et al., 2001)

Fluid catalytic cracking is the most important conversion process of heavy crude oils into more valuable gasoline and lighter products in petroleum refineries. Main fractionator is a distillation column where are distilled reaction products vapors from reactor into the FCC end products of wet gas, gasoline and LCO (diesel), see Fig. 16. A detail model of fractionator has been developed and fitted to real system data (Cong,Yuan,Shan,1998).

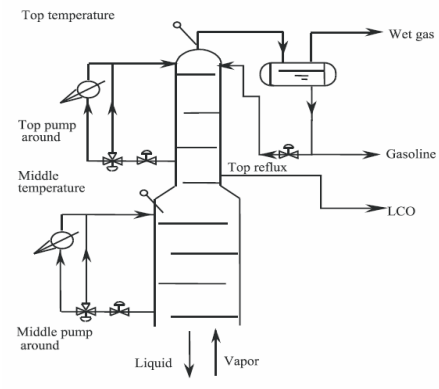

Fig. 15 Top part of a FCCU fractionator

Following linear system was derived by discretization and linearization from this plant around a nominal operating point.

$$x_{k+N} = 0.01\begin{bmatrix} 90.9028 & -6.4632 & 5.7545 & 1.5616 \\ -1.2803 & 921012 & 1.5499 & 3.3564 \\ -0.7713 & -4.2529 & 85.3340 & 3.4314 \\ 0.2413 & -0.7122 & -0.1415 & 856537 \end{bmatrix} x_k + \\ +0.01\begin{bmatrix} -3.6674 & -124077 & -0.0648 & -5.7626 & 0.1049 \\ -2.0582 & -0.5071 & 0.1088 & -7.2823 & 0.2324 \\ -1.3759 & -1.6691 & 1.5177 & -1.8955 & -0.1224 \\ 0.1182 & -0.0777 & -0.0002 & -0.3235 & 1.4456 \end{bmatrix} u_k \quad (68)$$

Upper and lower bounds, nominal point and legend for states and inputs are given in Tab.1

Tab. 1 States and control inputs description

Var	Description	Unit	Nominal operat. point		Lower bounds		Upper bounds	
x_1	Top vapor temperature	°C	x_1^{nom}	107	$\underline{x_1}$	106	$\overline{x^1}$	108
x_2	Middle vapor temperature	°C	x_2^{nom}	219	$\underline{x_2}$	218	$\overline{x^2}$	221
x_3	Top heat exchanger outlet temp.	°C	x_3^{nom}	87	$\underline{x_3}$	83	$\overline{x^3}$	91
x_4	Middle heat exchanger outlet temp.	°C	x_4^{nom}	199	$\underline{x_4}$	195	$\overline{x^4}$	203
u_1	Top pump-around tee valve position	%	u_1^{nom}	62.3	$\underline{u_1}$	0	$\overline{u^1}$	100
u_2	Top reflux valve position	%	u_2^{nom}	0	$\underline{u_2}$	0	$\overline{u^2}$	100
u_3	Top pump-around valve position	%	u_3^{nom}	50	$\underline{u_3}$	40	$\overline{u^3}$	80
u_4	Middle pump-around tee valve position	%	u_4^{nom}	67.9	$\underline{u_4}$	0	$\overline{u^4}$	100
u_5	Middle pump-around valve position	%	u_5^{nom}	50	$\underline{u_5}$	40	$\overline{u^5}$	80

States and inputs constraints of linear model are calculated from this table

$$\begin{aligned} &\begin{bmatrix}-1 & -1 & -4 & -4\end{bmatrix}^T \le x_k \le \begin{bmatrix}1 & 2 & 4 & 4\end{bmatrix}^T \\ &\begin{bmatrix}-62.3 & 0 & -10 & -67.9 & -10\end{bmatrix}^T \le u_k \le \begin{bmatrix}37.7 & 100 & 30 & 32.1 & 30\end{bmatrix}^T \end{aligned} \tag{69}$$

Penalty matrices and initial state for MPC implementation are $Q = I, R = 0$ and

$x_0 = \begin{bmatrix}-1 & 2 & -4 & 4\end{bmatrix}^T$

As obviously solving LMI (33) leads to terminal cost matrix P

$$P = \begin{bmatrix} 6.0311 & -2.1527 & 0.6344 & 0.0666 \\ -2.1527 & 9.2189 & -0.9565 & 0.5327 \\ 0.6344 & -0.9565 & 5.0739 & 0.5141 \\ 0.0666 & 0.5327 & 0.5141 & 5.4089 \end{bmatrix} \tag{70}$$

Due to the fact that we have 5 inputs and only 4 states is not possible to make explicit solution of stabilizing intervals for each input. The method of constructing polytopic sub-set of stabilizing controls, which is derived in section *4.2* has to be implemented. Next the input constraints polytopic set is constructed and the intersection between these two sets is detected. The first prediction horizon with non-empty intersection is the shortest stabilizing and feasible horizon. By the reason that dimension of these sets is 5 is not easy to make their graphic representation.

After constructing intersections for first 20 horizons was found out that horizon equal $N = 2$ is the shortest one. To confirm it, have a look to Fig. 16 where is depicted the intersection between input constraints and stabilizing controls set. In Fig. 17 are plotted the histories of states and inputs.

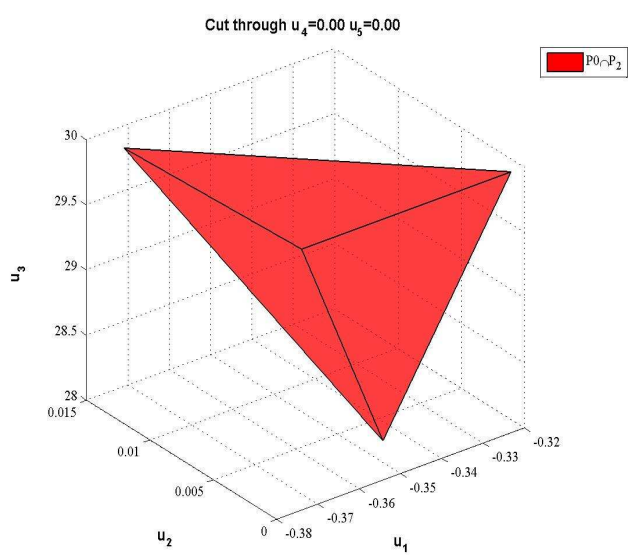

Fig. 16 Intersection between input constraint set U and sub-set of stabilizing controls for N=2.

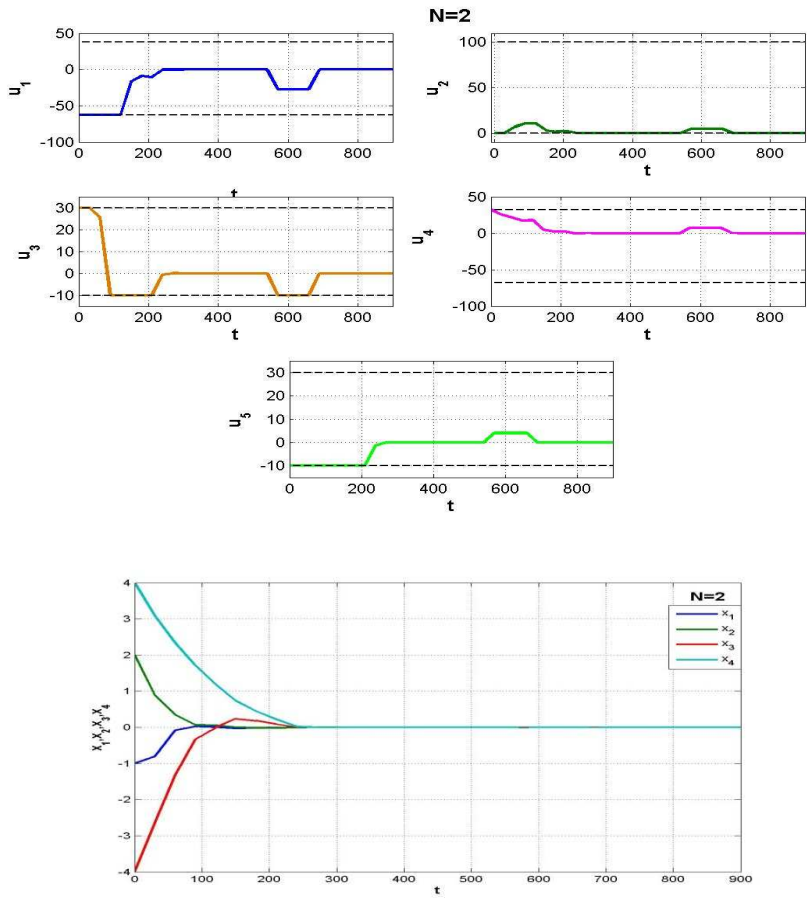

Fig. 17 State and input histories for plant (68) in closed loop with the introduced predictive control algorithm.

Owing to comparison, invariant terminal set is constructed.

$$K = \begin{bmatrix} 0.1481 & 0.0712 & 0.0688 & 0.0186 \\ 0.6175 & -0.2619 & 0.1570 & 0.0253 \\ -0.0011 & 0.0066 & -0.0644 & -0.0096 \\ 0.1506 & 0.4544 & 0.0666 & 0.0781 \\ -0.0006 & -0.0240 & 0 & -.0686 \end{bmatrix} \tag{71}$$

Ensuring stability by terminal matrix P and terminal invariant set caused enlarging stabilizing horizon to 3. There are not evident differences in states histories between

running MPC without and with terminal set. Inputs in both cases perform almost the same (see Fig. 18).

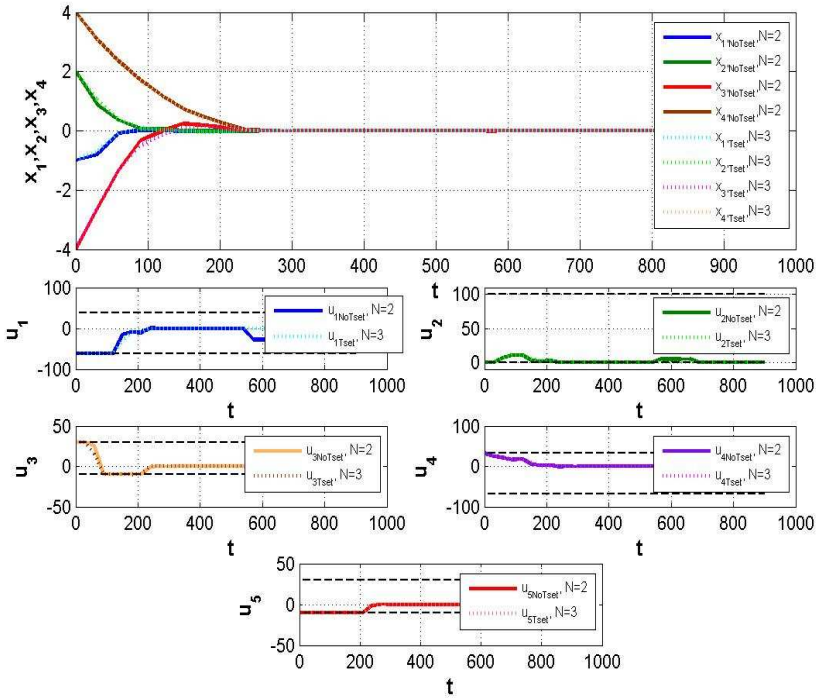

Fig. 18 Comparison performance of states and inputs for plant (68) with and without terminal set in MPC algorithm.

6 CONCLUSION

The thesis proposed a novel MPC method for guarantying asymptotic stability and feasibility for constrained linear systems. Terminal matrix P is chosen as solution of LMI and ensures that set of stabilizing controls is non-empty. It guarantees stability of the systems when constraints are not presented for any prediction horizon. In case when input and state constraints are active this method enables to determine the shortest feasible horizon just by explicit checking the intersection between input constraints set and set of stabilizing controls. This thesis also showed the procedures how to calculate the sets of stabilizing controls for SISO and MIMO systems. All simulations approve the results and advantages against to MPC with terminal constraints. The prediction horizon that presented approach requires will always be less than or equal to that required in controller with terminal constraints. In some simulations there will be a significant reduction in the length of the horizon as shown in the results of this thesis. Other important benefit that this stabilizing method offers is that this MPC strategy provide control which is at least as good as strategy of terminal matrix and terminal constraints. Further work is now required to make more detail investigation about infinity norm and to make extension of approach for nonlinear and hybrid systems.

7 RESUMÉ

Diplomová práca analyzuje stabilitu lineárnych systémov s ohraničeniami stavov a vstupov, keď riadenie je zabezpečené pomocou prediktívneho riadenia. V súčasnosti je štandardný spôsob garantovania stability založený na definovaní terminálnej invariantnej množiny, ktorá je implementované do optimalizačného problému prediktívneho riadenia a na penalizácii koncového stavu. V práci sa prezentuje prístup ku riadeniu, ktorý garantuje stabilitu bez použitia koncových ohraničení, čo značne redukuje výpočtovú náročnosť. Základom je existencia množiny stabilizujúcich vstupov. Sú to také vstupy, ktoré garantujú stabilitu v Ljapunovovom zmysle, čiže zabezpečujú klesanie účelovej funkcie (4). Táto množina je funkciou posledného predikovaného stavu a teda závisí aj od dĺžky predikčného horizontu. Zo závislosti tejto množiny stabilizujúcich vstupov od predikčného horizontu vieme determinovať najkratší stabilizujúci horizont. Je to prvý, pre ktorý existuje neprázdny prienik medzi spomínanou množinou stabilizujúcich vstupov a vstupnými ohraničeniami (18). Vtedy platí, že pre danú predikčnú dĺžku existuje aspoň jeden vstup, ktorý je stabilizujúci a zároveň uskutočniteľný, pretože spĺňa ohraničenia kladené na vstupnú veličinu. Prediktívne riadenie je forma riadenia, kde je model riadeného systému použitý na určenie hodnoty vstupnej veličiny na základe predikcií účinkov daného zásahu na riadený výstup. Otázka stability je principiálnou témou každého riadenia, pretože je nepredstaviteľné, že by dynamický systém vykonával svoju činnosť spoľahlivo bez toho, aby bol stabilný. Iba stabilizujúce regulátory môžu byť implementované v praxi a priemysle. Teória stability vychádza z Ljapunovovej teórie a základom je implementácia účelovej funkcie ako Ljapunovovej funkcie. Vtedy je vhodnou voľbou terminálnej matice P zabezpečená stabilita pre neohraničené systémy a pre každú dĺžku predikčného horizontu. V súčasnosti sa každý systém musí vysporiadať s ohraničeniami stavových a vstupných veličín. Garancia stability je preto spojená s koncovými ohraničenia, ktoré definujú terminálnu množinu, ktorá musí byť invariantná. Cieľom riadenia je dostať posledný predikovaný stav do tejto množiny, v ktorej na princípe invariantnosti zostane už aj nasledujúci a každý ďalší stav. Pri vstupe systému do tejto množiny často dochádza k prepnutiu z prediktívneho regulátora na lokálny spätnoväzbový. Takéto riadenie sa nazýva duálne riadenie. Prezentovaný prístup garancie stability nepredpokladá žiadne terminálne množiny, stabilita je zabezpečená len

pomocou vhodnej voľby terminálnej matice P a realizovateľnosť je garantovaná pomocou dostatočnej dĺžky predikčného horizontu. Bilineárna rovnica, ktorej riešením je koncová penalizačná matica P je odvodená na základe predpokladu klesania účelovej funkcie (21). Pretože bilineárne rovnice nie sú jednoducho riešiteľné, sú uskutočnené dve substitúcie a predstavené nové premenné a bilineárny problém je transformovaný do problému LMI, ktorý je riešený pomocou toolboxu YALMIP (33). Výsledkom riešenia LMI je nielen terminálna matica P ale aj lokálna spätnoväzbová matica K, ktorá je neskôr použitá na konštrukcia terminálnych invariantných množín, ktoré sú implementované v porovnávacích simuláciách, ktoré potvrdzujú výhodnosť odvodenej metódy garancie stability. Jadrom práce je definovanie množiny U_N^k a postupy jej analytického vyjadrenia pre SISO a MIMO systémy. Množina U_N^k, ako už bolo spomenuté, obsahuje všetky stabilizujúce a realizovateľné vstupy pre daný predikčný horizont. Pre SISO systémy je matematické vyjadrenie množiny U_N^k jednoduchou záležitosťou. Keď poznáme penalizačnú maticu P z dôkazu pre klesanie účelovej funkcie, dostávame triviálnu skalárnu kvadratickú funkciu s neznámou u_{k+N} (24) . Riešením tejto nerovnice sú všetky vstupy u_{k+N}, ktoré sa nachádzajú medzi koreňmi kvadratickej funkcie. Všetky tieto vstupy sú stabilizujúce, pretože zabezpečujú klesanie účelovej funkcie. Interval u_{k+N} závisí od posledného predikovaného stavu x_{k+N}, ktorý je známy a bol získaný riešením optimalizačného problém pre konkrétny predikčný horizont. Z tohto dôvodu môže byť zobrazená funkčná závislosť medzi intervalom stabilizujúcich vstupov u_{k+N} a dĺžkou predikčného horizontu. Pretože vstupy musia nielen garantovať stabilitu, ale byť aj realizovateľné, čiže spĺňať vstupné ohraničenia, množina U_N^k je definovaná ako prienik medzi intervalom stabilizujúcich vstupov u_{k+N} a intervalom vstupných ohraničení (28) . Prvý predikčný horizont pre ktorý je množina U_N^k neprázdna, je najkratším horizontom, ktorý garantuje stabilitu a realizovateľnosť. Celá táto procedúra môže byť realizovaná off-line. Pre sumarizáciu je postup determinovania najkratšieho predikčného horizontu nasledujúci. Pre každú dĺžku predikčného horizontu sa vypočíta posledný predikovaný stav. Ten je následný použitý do skalárnej kvadratickej funkcie (24) odkiaľ sa vypočíta interval stabilizujúcich vstupov u_{k+N}. Potom sa urobí

prienik tohto interval s intervalom vstupných ohraničení a získa sa tak množina U_N^k (28) . Prvý predikčný horizont garantujúci neprázdnosť množiny U_N^k je najkratším stabilizujúcim a realizovateľným horizontom. Pre MIMO systémy je konštrukcia množiny U_N^k zložitejšia, pretože nerovnica pre u_{k+N} (24) už nie je skalárnou kvadratickou funkciou a to v dôsledku, že u_{k+N} je vektor. Pre účelovú funkciu MIMO systémov sa predpokladá, že penalizačná matica R je rovná nule, čo výrazne zjednoduchší odvodzovanie vzťahov pre množinu U_N^k . Dôležitým nástrojom je existencia Choleského dekompozícií pre matice P a $P-Q$. Pri týchto predpokladoch môže byť nerovnica (28) pre SISO systém pozmenená do nerovnice (42) pre MIMO systém. Táto sústava nerovníc je jednoduchými úpravami prepísaná do výrazu pre polytopickú množinu (46). Tento polytop reprezentuje podmnožinu stabilizujúcich vstupov, čiže takých, ktoré garantujú klesanie účelovej funkcie. Na rozdiel od SISO systémov nedostávame interval stabilizujúcich vstupov, ale podmnožinu stabilizujúcich vstupov. Množina U_N^k je potom definovaná ako prienik medzi touto podmnožinou a množinou vstupných ohraničení (47). Najkratší interval garantujúci stabilitu a realizovateľnosť je odvodený znova z funkčnej závislosti od dĺžky predikčného horizontu a je prvý, pre ktorý je množina U_N^k neprázdna. Na prezentáciu výhod predloženého prístupu garancie stability bolo uskutočnených niekoľko simulácií na piatich príkladoch. Tri z nich sú príklady SISO systémov a dva MIMO systémov. Porovnávali sme so všeobecným prístupom garancie stability založenom nielen na penalizácií koncového stavu ale aj cieľovej množine. Cieľová invariantná množina bola konštruovaná pomocou lokálnej spätnoväzbovej matice K, ktorá je súčasťou riešenia LMI pre terminálnu penalizačnú maticu P (33). Všetky príklady potvrdzujú výhody opísaného prístupu garancie stability. Je dosiahnutá taká istá alebo podobná kvalita riadenia ako pomocou metódy, ktorá do optimalizačného problému včleňuje ohraničenia koncového stavu. Navyše najkratší horizont zabezpečujúci stabilitu a realizovateľnosť je v danom prístupe vždy kratší ako dĺžka predikčného horizontu v prístupe s terminálnymi ohraničeniami. Ďalšie výzvy prístupu garancie stability sa týkajú nekonečno noriem a nelineárnych systémov ako aj matematického dôkazu neprázdnosti množiny U_N^k po celý čas riadenia.

REFERENCES

[1] BEMPORAD, A. - MORARI, M. - Dua, V., - Pistikopoulos, E. N.: The explicit linear quadratic regulator for constrained systems. *Automatica*, vol.38, pp. 3-20, 2002

[2] KOTHARE, V. M. - BALAKRISHNAN, V. - MORARI, M.: Robust Constrained Model Predictive Control using Linear Matrix Inequality. *American Control Conference Baltimore Maryland*, 1994

[3] LAZAR, M.: Model Predictive Control of Hybrid Systems: Stability and Robustness, Eindhoven : Technische Universiteit Eindhoven, 2006

[4] LIMON, D. - ALAMO, T. - Salas, F. - CAMACHO, E. F.: On the Stability of Constrained MPC Without Terminal Constraints. *IEEE Transactions on Automatic Control* , vol. 51, no.5, 2006

[5] MARJANOVIC, O. - LENNOX. B.: Infinite horizon model predictive control with no terminal constraints. *Computers and Chemical Engineering*, vol.28, pp. 2605-2610, 2004

[6] MAYNE, D. Q. - DONA, J. A. - GOODWIN, G. C.: Improved stabilizing conditions for model predictive control. *IEEE Conference on Decision and Control*, *Sydney*, 2000

[7] MAYNE, D. Q. - RAWLINGS, J. B. - RAO, C. V. - SCOKAERT, P. O. M. : Constrained model predictive control: Stability and optimality. *Automatica*, vol. 36, pp. 789-814, 2000

[8] RAUOVA, I. - KVASNICA, M. - CIRKA, L. - FIKAR, M.: Real- time model predictive control of a laboratory liquid tanks system. *International Technical Computing Conf. Prague 2009*

[9] VADA, J. - SLUPPHAUG, O. - JOHANSEN, T.A. - FOSS, B. A.: Linear MPC with optimal prioritized infeasibility handling : application, computational issues and stability. *Automatica*, vol. 37, pp. 1835-1843, 2001

[10] CONG, S. B. – YUAN, P. – SHEN, F.: An integrated non-equilibrium dynamic model of petroleum distillation column, unpublished manuscript, 1998

[11] CHRISTOPHERSEN, F.J. – MORARI, M.: Further results on infinity norms as Lyapunov function for linear systems. *IEEE Transactions on Automatic Control* , vol. 52, no.3, 2007

[12] LAZAR, M. – HEEMELS, W.P.M.H. – WEILAND, S. – BEMPORAD, A. – PASTRAVANU, O.: Infinity norms as Lyapunov functions for model predictive control of constrained PWA systems. Lecture Notes in Computer Science, vol. 3414, pp 417-432, Springer-Verlag

[13] MURRAY, R.M. – LI, Z. – SASTRY, S.S.: Lyapunov stability theory http://www.cds.caltech.edu/~murray/courses/primer-f01/mls-lyap.pdf