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ANISOTROPIC BALANCED TRUNCATION — APPLICATION TO REDUCED-ORDER CONTROLLER DESIGN

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Abstract: This paper addresses the problem of reduced-order normalized anisotropic optimal controller design by anisotropic balanced truncation. This controller is the solution to the normalized anisotropy-based stochastic $\mathcal{H}_\infty$ problem. Anisotropic balanced truncation is aimed at reducing the order of closed-loop system. Two respective Riccati equations involved are used to define a set of closed-loop input-output invariants of closed-loop system called anisotropic characteristic values. The part of controller corresponding to smaller anisotropic characteristic values is truncated to give a reduced-order one. Truncation is carried out for the closed-loop state-space realization in anisotropic balanced coordinates, when the product of two respective Riccati equation solutions is a diagonal matrix with the squares of anisotropic characteristic values situated in descending order on its main diagonal. In anisotropic balanced coordinates, small characteristic values correspond to states which are easy to filter and control in a sense of anisotropic norm. It is shown that the reduced-order anisotropic controller is the full-order one for the reduced-order plant. An example of application to flight control in a windshear is given.

Keywords: Stochastic norm, information, order reduction

1. INTRODUCTION

The stochastic approach to $\mathcal{H}_\infty$ optimization introduced by Vladimirov et al. (1996-1), (1996-2) is based on using the anisotropic norm of a system as performance criterion. The anisotropic norm being a special case of stochastic norm is a quantitative index of system sensitivity to random input disturbances with mean anisotropy bounded by known nonnegative parameter. In turn, the mean anisotropy of a vector random sequence produced by a stable generating filter from vector zero-mean Gaussian white noise with scalar covariance matrix is a measure of colouredness of this sequence, that is a measure of correlation of vector components of the sequence (spatial part of the mean anisotropy), as well as a measure of correlation of different elements of this sequence (temporal part of the mean anisotropy). The latter coincides with the (mutual) information about an element of the sequence contained in the past history of this sequence. It has been shown that $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of a linear discrete time-invariant system are two limiting cases of the anisotropic norm as the mean anisotropy level of input random disturbance tends to zero or infinity, respectively. Therefore, this approach combines the attractive features of robust control and information theories holding an intermediate position between $\mathcal{H}_2$/LQG and $\mathcal{H}_\infty$ problems. Given a standard plant model and input mean anisotropy level, the anisotropy-based stochastic $\mathcal{H}_\infty$ problem consists in finding an output-feedback dynamic controller to internally stabilize the closed-loop system and minimize its anisotropic norm. The solution to this problem presented by Vladimirov et al. (1996-2) yields to the full-order controller and results in solving a cross-coupled nonlinear algebraic equation system defining the controller state-space realization matrices. However, we are interested in obtaining a reduced-order anisotropic controller.

The approximative approach to model reduction according to minimum anisotropic norm performance was introduced by Kurdyukov and Tchaikovsky (2008). A reduced-order model obtained by this method approximates the behaviour of a full-order system in steady-state mode, but it does not reflect the full-order system dynamics, since does not take into account pole locations of full- and reduced-order systems at all. Besides that, this method is intended for an open-loop system, therefore it accounts for neither controller properties nor even controller presence. This paper addresses the problem of reduced-order normalized anisotropic optimal controller design by means of anisotropic balanced truncation, which is close to LQG and $\mathcal{H}_\infty$ balanced truncation techniques introduced by Jonkhere and Silverman (1983), Mustafa and Glover (1991), correspondingly, and aimed at reducing the order of a closed-loop system. Two respective Riccati equations involved are used to define a set of closed-loop input-output system invariants called the *anisotropic characteristic values*. The part of the plant or controller corresponding to smaller anisotropic characteristic values is truncated to give a reduced-order plant or controller. Truncation is carried out for the closed-loop realization in anisotropic balanced coordinates, when the product of two respective Riccati equation solutions is a diagonal matrix with the squares of anisotropic characteristic values situated in descending order on its main diagonal. In anisotropic balanced coordinates, small characteristic values correspond to states which are easy to filter and control in a sense of anisotropic norm. It will be shown that the reduced-order controller is the full-order one for the reduced-order plant.

The paper structure is as follows. In Section 2 we consider the normalized anisotropy-based $\mathcal{H}_\infty$ problem. Subsection 2.1 introduces the problem statement together with some necessary background. The state-space structure of the worst-case input generating filter together with a sufficient saddle-point type condition for optimality of a controller in problem are given in Subsection 2.2. The algebraic equation system defining the state-estimating optimal controller is introduced in Subsection 2.3. The technique of controller order reduction by anisotropic balanced truncation is considered in Section 3. The notion of anisotropic characteristic values is introduced in Subsection 3.1 together with a nonsingular similarity transformation putting the system realization into the anisotropic balanced coordinates. Subsection 3.2 represents the expressions for state-space realizations of reduced-order plant and anisotropic controller. Section 4 is devoted to an example of reduced-order anisotropic controller design for longitudinal flight control in a windshear.

2. NORMALIZED ANISOTROPY-BASED STOCHASTIC $\mathcal{H}_\infty$ OPTIMIZATION PROBLEM

The normalized anisotropy-based stochastic $\mathcal{H}_\infty$ problem is characterized by some features making it different from the general-case problem considered by Vladimirov et al. (1996-2). To disclose these important distinctions, it is preferable to consider in details the statement and solution of the normalized problem.

2.1 *Problem Statement*

All the encountered random elements are assumed to be defined on a complete probability space $(\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$ with the set $\mathbf{\Omega}$ of primary outcomes, the σ-algebra $\mathcal{F}$ of random events, and the probability measure $\mathbf{P}$ with the corresponding expectation functional $\mathbf{E}$.

Consider a linear discrete time-invariant causal plant $P(z)$ with n-dimensional internal state X, m_1-dimensional disturbance input W, m_2-dimensional control input U, p_1-dimensional controlled output Z, and p_2-dimensional measured output Y. All these signals are double-sided discrete-time sequences related to each other by the equations

$$\begin{bmatrix} x_{k+1} \\ \hline z_k \\ y_k \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \begin{bmatrix} x_k \\ \hline w_k \\ u_k \end{bmatrix}, \qquad (1)$$

where all matrices have appropriate dimensions, $p_1 = m_1 = p_2 + m_2$, and the matrices

$$B_1 = \begin{bmatrix} B_2 & 0 \end{bmatrix}, \; D_{21} = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix},$$
$$C_1 = \begin{bmatrix} C_2 \\ 0 \end{bmatrix}, \; D_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}. \qquad (2)$$

The state-space realization (A, B_2, C_2) is assumed to be minimal (i.e. (A, B_2) is controllable, (A, C_2) is observable). Plant (1) is called the normalized standard plant. It is easily seen that the normalized standard plant has the controlled output

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} C_2 X \\ U \end{bmatrix} \qquad (3)$$

and the disturbance input

$$W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \qquad (4)$$

partitioned into m_2-dimensional block W_1 and p_2-dimensional block W_2 such that W_1 enters the system together with the control signal (actuator noise) while W_2 affects upon the measured output Y (measurement noise).

The only prior information on the probability distribution of the random external disturbance W consists in the following: W is a stationary Gaussian sequence with mean anisotropy bounded by a known nonnegative parameter α. Specifically, the latter means that W is produced from m_1-dimensional Gaussian white noise V with zero mean and identity covariance matrix: $\mathbf{E}(v_k) = 0$, $\mathbf{E}(v_k v_k^{\mathrm{T}}) = I_{m_1}$, $-\infty < k < +\infty$, by an unknown generating filter G in the family

$$\mathcal{G}_\alpha \triangleq \left\{ G \in \mathcal{H}_2^{m_1 \times m_1} : \ \overline{\mathbf{A}}(G) \leqslant \alpha \right\},$$

where

$$\overline{\mathbf{A}}(G) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln\det \left\{ \frac{m_1}{\|G\|_2^2} \widehat{G}(\omega)\widehat{G}^*(\omega) \right\} d\omega$$

is the mean anisotropy functional introduced by Vladimirov et al. (1996-1) (also called the mean anisotropy), the angular boundary value

$$\widehat{G}(\omega) \triangleq \lim_{r \to 1-0} G(re^{i\omega}),$$

and

$$\|G\|_2 \doteq \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{tr}\{\widehat{G}(\omega)\widehat{G}^*(\omega)\} d\omega \right\}^{1/2}.$$

At that, no assumption is made in respect of cross-correlation of blocks W_1 and W_2.

The normalized anisotropy-based stochastic $\mathcal{H}_\infty$ optimization problem is formulated as follows.

Problem 1. Given normalized standard plant (1) and input mean anisotropy level $\alpha \geqslant 0$, find a strictly causal controller K to internally stabilize the closed-loop system $F(z)$ given by the lower linear-fractional transformation of the pair (P, K) :

$$F(z) = \mathcal{F}_l(P, K) = P_{11} + P_{12}K(I_{p_2} - P_{22}K)^{-1}P_{21}, \tag{5}$$

where

$$P_{ij}(z) \sim \left[\begin{array}{c|c} A & B_j \\ \hline C_i & D_{ij} \end{array} \right], \quad i, j = \overline{1, 2}, \tag{6}$$

and minimize its α-anisotropic norm:

$$|\!|\!| F |\!|\!|_\alpha \doteq \sup_{G \in \mathcal{G}_\alpha} \frac{\|FG\|_2}{\|G\|_2} \to \inf_K, \quad K \in \mathcal{K}. \tag{7}$$

The formulated problem (just as any of minimax problems) can be considered as an antagonistic game of two opponents, control system designer and nature. The set of designer's strategies in this game is the set $\mathcal{K}$ of internally stabilizing controllers, and the set of nature's strategies is the family $\mathcal{G}_\alpha$ of filters generating random sequences with mean anisotropy bounded by known parameter $\alpha \geqslant 0$.

Denote that in the case of zero mean anisotropy level $\alpha = 0$ the formulated above problem coincides with the normalized LQG problem considered by Jonkhere and Silverman (1983), Mustafa and Glover (1991).

2.2 *Worst-Case Generating Filter for Closed-Loop System*

Since problem (7) is a minimax problem, one can use the results of differential game theory to formulate a saddle-point type condition of optimality. For any generating filter $G \in \mathcal{G}_\alpha$ and any internally stabilizing controller $K \in \mathcal{K}$, let us introduce the following sets

$$\mathcal{K}_\alpha^\diamond(G) \triangleq \operatorname*{Arg\,min}_{K \in \mathcal{K}} \|FG\|_2, \quad G \in \mathcal{G}_\alpha, \tag{8}$$

$$\mathcal{G}_\alpha^\diamond(K) \triangleq \operatorname*{Arg\,max}_{G \in \mathcal{G}_\alpha} \frac{\|FG\|_2}{\|G\|_2}, \quad K \in \mathcal{K}. \tag{9}$$

These sets are assumed to be nonempty. Set (8) consists of the controllers being solutions to the weighted LQG problem under the assumption that the closed-loop system input is fed with coloured noise $W = GV$. Any such controller $K \in \mathcal{K}_\alpha^\diamond(G)$ minimizes variance of the output random sequence Z (LQG-cost)

$$J_{\mathrm{LQG}}(FG) \triangleq \lim_{N \to \infty} \mathbf{E} \left(\frac{1}{2N} \sum_{k=-N}^{N} z_k^{\mathrm{T}} z_k \right)$$

$$= \lim_{N \to \infty} \mathbf{E} \left(\frac{1}{2N} \sum_{k=-N}^{N} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} C_2^{\mathrm{T}} C_2 & 0 \\ 0 & I_{m_2} \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \right) \tag{10}$$

of weighted closed-loop system FG with input disturbance V.

In turn, set (9) is formed by the filters generating Gaussian random sequences W with spectral densities

$$\widehat{S}_{WW}(\omega) = \widehat{G}(\omega)\widehat{G}^*(\omega), \quad \omega \in [-\pi, \pi],$$

which are the worst (i.e. the most adverse) for the closed-loop system $F = \mathcal{F}_l(P, K)$. Although the set $\mathcal{G}_\alpha^\diamond(K)$ is invariant under right-hand multiplication by an all-pass system, and hence, consists of infinite number of filters, all of them generate the sequences with the unique up to scalar multiplier worst-case spectral density (see Vladimirov et al. (1996-1)). Such filters are called the worst-case input generating filters.

Thus the relation

$$(\mathcal{K}_\alpha^\diamond \circ \mathcal{G}_\alpha^\diamond)(K) \triangleq \bigcup_{G \in \mathcal{G}_\alpha^\diamond(K)} \mathcal{K}_\alpha^\diamond(G), \quad K \in \mathcal{K}$$

defines the (generally ambiguous) composition $\mathcal{K}_\alpha^\diamond \circ \mathcal{G}_\alpha^\diamond : \mathcal{K} \to 2^{\mathcal{K}}$ of the mappings $\mathcal{K}_\alpha^\diamond : \mathcal{G}_\alpha \to 2^{\mathcal{K}}$

and $\mathcal{G}_\alpha^\diamond \colon \mathcal{K} \to 2^{\mathcal{G}_\alpha}$. The following lemma that can be proved similar to Lemma 1 in Vladimirov et al. (1996-2) establishes a sufficient saddle-point type condition of optimality for problem (7).

Lemma 1. If a controller K is a stationary point of the mapping $\mathcal{K}_\alpha^\diamond \circ \mathcal{G}_\alpha^\diamond$, that is, if there exists a generating filter G such that

$$K \in \mathcal{K}_\alpha^\diamond(G), \quad G \in \mathcal{G}_\alpha^\diamond(K),$$

then the controller K is a solution to problem (7).

Let $K \in \mathcal{K}$ be an admissible controller with n-dimensional internal state H related with the measurement Y and control sequence U by the equations

$$\begin{bmatrix} h_{k+1} \\ \hline u_k \end{bmatrix} = \left[\begin{array}{c|c} A_{\rm c} & B_{\rm c} \\ \hline C_{\rm c} & 0 \end{array}\right] \begin{bmatrix} h_k \\ \hline y_k \end{bmatrix}, \tag{11}$$

where $A_{\rm c}, B_{\rm c}, C_{\rm c}$ are constant matrices of appropriate dimensions. Then the state-space realization of closed-loop system (5) is given by

$$F(z) \sim \left[\begin{array}{c|c} A_{\rm cl} & B_{\rm cl} \\ \hline C_{\rm cl} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A & B_2C_{\rm c} & B_1 \\ B_{\rm c}C_2 & A_{\rm c} & B_{\rm c}D_{21} \\ \hline C_1 & D_{12}C_{\rm c} & 0 \end{array}\right] \tag{12}$$

with $A_{\rm cl}$ stable, that is, taking into account (2)–(4), input W and output Z of the closed-loop system F are related by the equations

$$\begin{bmatrix} x_{k+1} \\ h_{k+1} \\ \hline z_{1k} \\ z_{2k} \end{bmatrix} = \left[\begin{array}{cc|cc} A & B_2C_{\rm c} & B_2 & 0 \\ B_{\rm c}C_2 & A_{\rm c} & 0 & B_{\rm c} \\ \hline C_2 & 0 & 0 & 0 \\ 0 & C_{\rm c} & 0 & 0 \end{array}\right] \begin{bmatrix} x_k \\ h_k \\ \hline w_{1k} \\ w_{2k} \end{bmatrix}.$$

It is assumed that system $F(z)$ satisfies strict inequality

$$\frac{1}{\sqrt{m_1}}\|F\|_2 < \|F\|_\infty, \tag{13}$$

otherwise, its anisotropic norm coincides trivially with the scaled $\mathcal{H}_2$ norm. It should be noted that inequality (13) is violated iff the closed-loop system F is inner up to a nonzero constant factor, in which case there exists a number $\lambda > 0$ such that $\widehat{F}^*(\omega)\widehat{F}(\omega) = \lambda I_{m_1}$ for almost all $\omega \in [-\pi, \pi]$ (Vladimirov et al. (1996-1)).

Lemma 2. Let the realization (A, B_2, C_2) of plant (1) be minimal, and let the closed-loop system F not be inner. Then, for any controller $K \in \mathcal{K}$ and given level of input mean anisotropy $\alpha \geqslant 0$, there exists the unique pair (q, R) of the scalar parameter $q \in [0, \|F\|_\infty^{-2})$ and stabilizing solution $R = R^{\rm T} \geqslant 0$ of the algebraic Riccati equation

$$\left.\begin{array}{l} R = A_{\rm cl}^{\rm T}RA_{\rm cl} + qC_{\rm cl}^{\rm T}C_{\rm cl} + L^{\rm T}\Sigma^{-1}L \\ \Sigma \triangleq (I_{m_1} - B_{\rm cl}^{\rm T}RB_{\rm cl})^{-1} \\ L = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \triangleq \Sigma B_{\rm cl}^{\rm T}RA_{\rm cl} \end{array}\right\} \tag{14}$$

such that

$$-\frac{1}{2}\ln\det\left\{\frac{m_1\Sigma}{\operatorname{tr}(LP_cL^{\rm T} + \Sigma)}\right\} = \alpha, \tag{15}$$

where $P_c = P_c^{\rm T} > 0$ is the controllability gramian of the generating filter

$$G(z) \sim \left[\begin{array}{c|c} A_{\rm cl} + B_{\rm cl}L & B_{\rm cl}\Sigma^{1/2} \\ \hline L & \Sigma^{1/2} \end{array}\right] \tag{16}$$

satisfying the Lyapunov equation

$$P_c = (A_{\rm cl}+B_{\rm cl}L)^{\rm T}P_c(A_{\rm cl}+B_{\rm cl}L)+B_{\rm cl}\Sigma B_{\rm cl}^{\rm T}. \tag{17}$$

At that, the filter (16) is a representative of set (9) of the worst-case input generating filters satisfying factorization

$$\widehat{G}(\omega)\widehat{G}^*(\omega) = (I_{m_1} - q\widehat{F}^*(\omega)\widehat{F}(\omega))^{-1}.$$

Proof of this lemma follows immediately from Theorem 2 in Vladimirov et al. (1996-1) applied to closed-loop system (12).

Remark 1. Recall that a solution $R = R^{\rm T} \in \mathbb{R}^{2n\times 2n}$ of algebraic Riccati equation (14) is called stabilizing if the matrix $A_{\rm cl} + B_{\rm cl}L$ is stable and $\Sigma > 0$. For any controller $K \in \mathcal{K}$ and $q \in [0, \|F\|_\infty^{-2})$ equation (14) has the unique stabilizing solution and this solution is a positive semidefinite matrix (see Vladimirov et al. (1996-1)).

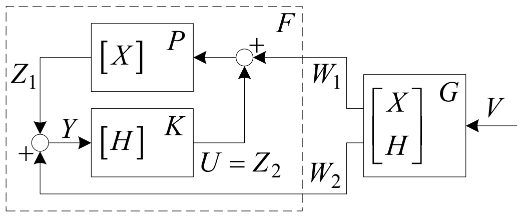

Fig. 1. Block diagram of weighted closed-loop system FG

Remark 2. The internal state of the worst-case generating filter G actually is a copy of the internal state of the closed-loop system F (see block diagram at Fig. 1). Thus, equations (1) and (11) combined with

$$w_k = L_1x_k + L_2h_k + \Sigma^{1/2}v_k$$

relates the input V, output $W = GV$, and internal state (X, H) of worst-case generating filter (16). Taking into account partitioning (4) of filter output W, one can put down the following equations

$$\begin{bmatrix} x_{k+1} \\ h_{k+1} \\ \hline w_{1k} \\ w_{2k} \end{bmatrix} = \left[\begin{array}{cc|c} A + B_2L_{11} & B_2(C_{\rm c} + L_{12}) & B_2\widetilde{\Sigma}_1 \\ B_{\rm c}(C_2 + L_{21}) & A_{\rm c} + B_{\rm c}L_{22} & B_{\rm c}\widetilde{\Sigma}_2 \\ \hline L_{11} & L_{22} & \widetilde{\Sigma}_1 \\ L_{21} & L_{22} & \widetilde{\Sigma}_2 \end{array}\right] \begin{bmatrix} x_k \\ h_k \\ \hline v_k \end{bmatrix}$$

defining the dynamics of the worst-case generating filter $G(z)$, where

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} L_1 & L_2 \end{bmatrix} = L, \tag{18}$$

$$\begin{bmatrix} \widetilde{\Sigma}_1 \\ \widetilde{\Sigma}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{\Sigma}_{11} & \widetilde{\Sigma}_{12} \\ \widetilde{\Sigma}_{21} & \widetilde{\Sigma}_{22} \end{bmatrix} = \Sigma^{1/2}. \tag{19}$$

For weighted closed-loop system FG we have

$$\begin{bmatrix} x_{k+1} \\ h_{k+1} \\ z_{1k} \\ z_{2k} \end{bmatrix} = \left[\begin{array}{cc|c} A + B_2 L_{11} & B_2(C_c + L_{12}) & B_2 \widetilde{\Sigma}_1 \\ B_c(C_2 + L_{21}) & A_c + B_c L_{22} & B_c \widetilde{\Sigma}_2 \\ \hline C_2 & 0 & 0 \\ 0 & C_c & 0 \end{array}\right] \begin{bmatrix} x_k \\ h_k \\ v_k \end{bmatrix}. \tag{20}$$

2.3 *Optimal State-Estimating Controller for Weighted LQG Problem*

Let us fix the worst-case input generating filter $G^\diamond \in \mathcal{G}_\alpha^\diamond$ defined by Lemma 2 and consider the weighted plant

$$P_G \triangleq \begin{bmatrix} P_{11}G^\diamond & P_{12} \\ P_{21}G^\diamond & P_{22} \end{bmatrix} \sim$$

$$\left[\begin{array}{ccc|cc} A & B_2 L_{11} & B_2 L_{12} & B_2 \widetilde{\Sigma}_1 & B_2 \\ 0 & A + B_2 L_{11} & B_2(C_c + L_{12}) & B_2 \widetilde{\Sigma}_1 & 0 \\ 0 & B_c(C_2 + L_{21}) & A_c + B_c L_{22} & B_c \widetilde{\Sigma}_2 & 0 \\ \hline C_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{m_2} \\ C_2 & L_{21} & L_{22} & \widetilde{\Sigma}_2 & 0 \end{array}\right], \tag{21}$$

where P_{ij} are defined by (6), with two inputs V (Gaussian white noise) and U, two outputs $Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$, Y, and ($3n$)-dimensional internal state $\begin{bmatrix} X \\ X^0 \\ H^0 \end{bmatrix}$, where $\begin{bmatrix} X^0 \\ H^0 \end{bmatrix}$ is the internal state of the worst-case generating filter $G^\diamond$ (Fig. 2).

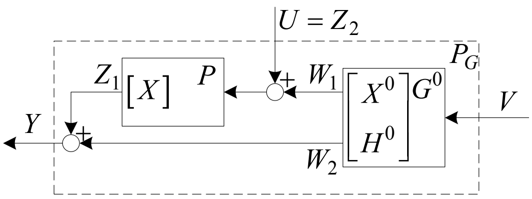

Fig. 2. Block diagram of plant P_G for weighted LQG problem

With the fixed worst-case input generating filter, Problem 1 is equivalent to the following weighted LQG problem for plant (21).

Problem 2. Given the weighted plant P_G, find a strictly causal controller K to internally stabilize closed-loop system $F_G = \mathcal{F}_l(P_G, K)$ and minimize LQG cost (10):

$$J_{\mathrm{LQG}}(F_G) \to \inf_{K \in \mathcal{K}}. \tag{22}$$

Since in general the internal state X in plant (1) is not measurable, and the measurement Y includes additive noise W_2, the desired controller K can be only the state-estimating output-feedback controller with the internal state H being the optimal mean-square estimate of the internal state X of plant (1).

Let $\mathcal{F}_k^Y$ denote the σ-algebra of random events induced by the history $(y_j)_{j \leqslant k}$ of the measurement signal Y at instant k, and $(\mathcal{F}_k^Y)_{-\infty<k<+\infty}$ be the flow of σ-algebras in $\mathcal{F}$ generated by the sequence Y.

Recall that stabilizing controller (11) is called state-estimating one if its n-dimensional internal state H coincides with the sequence of one-step predictors for the internal state X of plant (1) by the measurement signal Y under the worst-case input disturbance W, i.e. if

$$h_k = \mathbf{E}(x_k | \mathcal{F}_{k-1}^Y), \quad -\infty < k < +\infty,$$

when $W = GV$ with the worst-case input generating filter $G \in \mathcal{G}_\alpha^\diamond(K)$ (Vladimirov et al. (1996-2)).

Let us consider the weighted closed-loop system FG introduced by state-space equations (20) (Fig. 1). In this system, the following relations between the flows of σ-algebras generated by the stationary Gaussian sequences are valid:

$$\mathcal{F}_k^H \subset \mathcal{F}_{k-1}^Y \subset \mathcal{F}_{k-1}^V \supset \mathcal{F}_k^X, \quad -\infty < k < +\infty. \tag{23}$$

Denote by

$$\begin{aligned} \widehat{x}_k &\triangleq \mathbf{E}(x_k | \mathcal{F}_{k-1}^Y), \\ \widehat{\widehat{x}}_k &\triangleq \mathbf{E}(x_{k+1} | \mathcal{F}_{k-1}^Y), \\ \widehat{y}_k &\triangleq \mathbf{E}(y_k | \mathcal{F}_{k-1}^Y) \end{aligned} \tag{24}$$

the one-step and two-step predictors of the state X by observation Y, as well as the one-step self-predictor of the measurement Y. Predictors (24) are $\mathcal{F}_{k-1}^Y$-measurable, and the following prediction errors correspond to them:

$$\widetilde{x}_k \triangleq x_k - \widehat{x}_k, \ \widetilde{\widetilde{x}}_k \triangleq x_{k+1} - \widehat{\widehat{x}}_k, \ \widetilde{y}_k \triangleq y_k - \widehat{y}_k. \tag{25}$$

As it was noted by Vladimirov et al. (1996-2), the sequence of measurement prediction errors $(\widetilde{y}_k)_{-\infty<k<+\infty}$ is the martingale-difference (Liptser and Shiryaev (1977)) with respect to the flow $(\mathcal{F}_k^Y)_{-\infty<k<+\infty}$ and, hence, the zero-mean Gaussian white noise.

From equations (1), (2), (20) and inclusions (23), we have the following expressions for predictors (24) and prediction errors (25):

$$\left.\begin{aligned} \widehat{\widehat{x}}_k &= (A + B_2 L_{11}) \widehat{x}_k + B_2(C_c + L_{12}) h_k \\ \widehat{y}_k &= (C_2 + L_{21}) \widehat{x}_k + L_{22} h_k \end{aligned}\right\}, \tag{26}$$

$$\left.\begin{aligned} \widetilde{\widetilde{x}}_k &= (A + B_2 L_{11}) \widetilde{x}_k + B_2 \widetilde{\Sigma}_1 v_k \\ \widetilde{y}_k &= (C_2 + L_{21}) \widetilde{x}_k + \widetilde{\Sigma}_2 v_k \end{aligned}\right\}. \tag{27}$$

By virtue of Normal Correlation Lemma (see Liptser and Shiryaev (1977)), the predictor $\widehat{x}_{k+1}$ is given by

$$\widehat{x}_{k+1} = \widehat{\widehat{x}}_k + \mathbf{E}(\widetilde{\widetilde{x}}_k \widetilde{y}_k^{\mathrm{T}}) [\mathbf{E}(\widetilde{y}_k \widetilde{y}_k^{\mathrm{T}})]^{-1} \widetilde{y}_k, \tag{28}$$

whereas the covariance matrix of prediction error $\widetilde{x}_k$ is

$$\begin{aligned}\mathbf{E}(\widetilde{x}_k\widetilde{x}_k^{\mathrm{T}}) &= \mathbf{E}(\widetilde{\widetilde{x}}_k\widetilde{\widetilde{x}}_k^{\mathrm{T}}) \\ &\quad -\mathbf{E}(\widetilde{\widetilde{x}}_k\widetilde{y}_k^{\mathrm{T}})[\mathbf{E}(\widetilde{y}_k\widetilde{y}_k^{\mathrm{T}})]^{-1}\mathbf{E}(\widetilde{y}_k\widetilde{\widetilde{x}}_k^{\mathrm{T}}).\end{aligned} \tag{29}$$

Denoting

$$S \triangleq \mathbf{E}(\widetilde{x}_k\widetilde{x}_k^{\mathrm{T}}), \tag{30}$$

let us express the covariance matrices in (29) from equation (27) as follows:

$$\begin{aligned}\mathbf{E}(\widetilde{\widetilde{x}}_k\widetilde{\widetilde{x}}_k^{\mathrm{T}}) &= (A+B_2L_{11})S(A+B_2L_{11})^{\mathrm{T}} \\ &\quad +B_2\Sigma_{11}B_2^{\mathrm{T}}, \\ \mathbf{E}(\widetilde{y}_k\widetilde{y}_k^{\mathrm{T}}) &= (C_2+L_{21})S(C_2+L_{21})^{\mathrm{T}}+\Sigma_{22}, \\ \mathbf{E}(\widetilde{\widetilde{x}}_k\widetilde{y}_k^{\mathrm{T}}) &= (A+B_2L_{11})S(C_2+L_{21})^{\mathrm{T}}+B_2\Sigma_{12},\end{aligned} \tag{31}$$

where

$$\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{12}^{\mathrm{T}} & \Sigma_{22}\end{bmatrix} = \begin{bmatrix}\widetilde{\Sigma}_1\\ \widetilde{\Sigma}_2\end{bmatrix}\begin{bmatrix}\widetilde{\Sigma}_1^{\mathrm{T}} & \widetilde{\Sigma}_2^{\mathrm{T}}\end{bmatrix} = \Sigma. \tag{32}$$

Introducing the notation

$$\mathbf{E}(\widetilde{y}_k\widetilde{y}_k^{\mathrm{T}}) \triangleq \Theta, \quad \mathbf{E}(\widetilde{\widetilde{x}}_k\widetilde{y}_k^{\mathrm{T}}) \triangleq \Lambda\Theta, \tag{33}$$

from (29)–(31) we obtain the filtering algebraic Riccati equation

$$\left.\begin{aligned} S &= (A+B_2L_{11})S(A+B_2L_{11})^{\mathrm{T}} \\ &\quad +B_2\Sigma_{11}B_2^{\mathrm{T}}-\Lambda\Theta\Lambda^{\mathrm{T}} \\ \Theta &\triangleq (C_2+L_{21})S(C_2+L_{21})^{\mathrm{T}}+\Sigma_{22} \\ \Lambda &\triangleq ((A+B_2L_{11})S(C_2+L_{21})^{\mathrm{T}}+B_2\Sigma_{12})\Theta^{-1}\end{aligned}\right\} \tag{34}$$

in the prediction error covariance matrix S.

Remark 3. Denote that Riccati equation (34) has the unique stabilizing positive definite solution $S = S^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ such that the matrix $A+B_2L_{11}-\Lambda(C_2+L_{21})$ is stable (see Molinari (1975)).

Substituting (26) and (33) to (28) with

$$\widetilde{y}_k = y_k - (C_2+L_{21})\widehat{x}_k - L_{22}h_k$$

in mind, we obtain

$$\begin{aligned}\widehat{x}_{k+1} &= (A+B_2L_{11}-\Lambda(C_2+L_{21}))\widehat{x}_k \\ &\quad +(B_2(C_{\mathrm{c}}+L_{12})-\Lambda L_{22})h_k+\Lambda y_k.\end{aligned}$$

The last equation together with controller equations (11) shows that the sequence

$$\widehat{X} = (\widehat{x}_k)_{-\infty<k<+\infty}$$

is produced from the measurement Y by the system $E(z)$ (i.e. $\widehat{X} = EY$) with $2n$-dimensional internal state $\begin{bmatrix}\widehat{X}\\ H\end{bmatrix}$ and the state-space realization

$$E(z) \sim \left[\begin{array}{cc|c} A+B_2L_{11}-\Lambda(C_2+L_{21}) & B_2L_{12}+B_2C_{\mathrm{c}}-\Lambda L_{22} & \Lambda \\ 0 & A_{\mathrm{c}} & B_{\mathrm{c}} \\ \hline I_n & 0 & 0\end{array}\right]. \tag{35}$$

The following lemma defines the state-space realization matrices of the state-estimating controller.

Lemma 3. Let the state-space realization matrices of stabilizing controller (11) be given by

$$\begin{aligned} A_{\mathrm{c}} &= A+B_2(C_{\mathrm{c}}+M_1)-\Lambda(C_2+M_2), \\ B_{\mathrm{c}} &= \Lambda,\end{aligned} \tag{36}$$

where

$$M_1 \triangleq L_{11}+L_{12}, \quad M_2 \triangleq L_{21}+L_{22}, \tag{37}$$

and the matrix Λ is expressed via the stabilizing solution of filtering algebraic Riccati equation (34). Then controller (11) is the state-estimating one.

PROOF. Substituting (36) to (35) and applying Lemma 8 from Appendix, we obtain

$$\begin{aligned}E(z) &\sim \left[\begin{array}{c|c} A+B_2(C_{\mathrm{c}}+M_1)-\Lambda(C_2+M_2) & \Lambda \\ \hline I_n & 0\end{array}\right] \\ &= \left[\begin{array}{c|c} A_{\mathrm{c}} & B_{\mathrm{c}} \\ \hline I_n & 0\end{array}\right],\end{aligned}$$

i.e. the controller is the state-estimating one.

After designing the optimal state estimator, let us construct optimal estimate-feedback loop. By the state-estimating property of the controller K, the copy of its internal state H^0 coincides with the one-step predictor $\widehat{X}^0$ of the sequence X^0 :

$$h_k^0 \equiv \widehat{x}_k^0 \triangleq \mathbf{E}(x_k^0|\mathcal{F}_{k-1}^Y), \quad -\infty < k < +\infty. \tag{38}$$

Then the sequence $\widehat{X}^0$ defined by (38) and the sequence of the one-step predictors $\widehat{X}$ defined by (24) are governed by the equations

$$\left.\begin{aligned}\widehat{x}_{k+1} &= A\widehat{x}_k + B_2M_1\widehat{x}_k^0 + B_2u_k + \Lambda\widetilde{y}_k \\ \widehat{x}_{k+1}^0 &= (A+B_2(M_1+C_{\mathrm{c}}))\widehat{x}_k^0 + \Lambda\widetilde{y}_k\end{aligned}\right\}, \tag{39}$$

where

$$\widetilde{y}_k = y_k - C_2\widehat{x}_k - M_2\widehat{x}_k^0 \tag{40}$$

is the zero-mean Gaussian white noise with covariance matrix Θ defined by (34), the matrices M_1 and M_2 are given by (37). The one-step predictor of the output Z by the measurement Y is given by

$$\widehat{z}_k \triangleq \mathbf{E}(z_k|\mathcal{F}_{k-1}^Y) = \begin{bmatrix} C_2\widehat{x}_k \\ u_k\end{bmatrix}.$$

The covariance matrix of the corresponding prediction error $\widetilde{z}_k \triangleq z_k - \widehat{z}_k$ is

$$\mathbf{E}(\widetilde{z}_k\widetilde{z}_k^{\mathrm{T}}) = \begin{bmatrix} C_2SC_2^{\mathrm{T}} & 0 \\ 0 & 0\end{bmatrix},$$

where S is the stabilizing solution of filtering Riccati equation (34), and also does not depend on the controller matrices. It follows that problem (22) reduces to the state-feedback LQ problem of minimizing the LQ-cost

$$J_{\rm LQ} \triangleq \mathbf{E}\left(\sum_{k=0}^{+\infty} \widehat{z}_k^{\rm T}\widehat{z}_k\right) = \mathbf{E}\left(\sum_{k=0}^{+\infty}\begin{bmatrix}\widehat{x}_k\\ u_k\end{bmatrix}^{\rm T}\begin{bmatrix}C_2^{\rm T}C_2 & 0\\ 0 & I_{m_2}\end{bmatrix}\begin{bmatrix}\widehat{x}_k\\ u_k\end{bmatrix}\right) = \mathbf{E}\left(\sum_{k=0}^{+\infty}\begin{bmatrix}\widehat{x}_k\\ \widehat{x}_k^0\\ u_k\end{bmatrix}^{\rm T}\begin{bmatrix}C_2^{\rm T}C_2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & I_{m_2}\end{bmatrix}\begin{bmatrix}\widehat{x}_k\\ \widehat{x}_k^0\\ u_k\end{bmatrix}\right) \quad (41)$$

in the framework of dynamics (39) over stabilizing controllers $K \in \mathcal{K}$, which is solved standardly (see Dorato and Levis (1971)).

The optimal control law minimizing LQ-cost (41) is given by

$$u_k^\circ = N_1\widehat{x}_k + N_2\widehat{x}_k^0,$$

where the matrices N_1 and N_2 are expressed from the stabilizing solution $T_\star$ of the control algebraic Riccati equation

$$\left.\begin{aligned} T_\star &= A_\star^{\rm T}T_\star A_\star + C_\star^{\rm T}C_\star - N_\star^{\rm T}\Pi_\star N_\star\\ \Pi_\star &\triangleq B_\star^{\rm T}T_\star B_\star + I_{m_2}\\ N_\star &= \begin{bmatrix} N_1 & N_2\end{bmatrix} \triangleq -\Pi_\star^{-1}B_\star^{\rm T}T_\star A_\star\end{aligned}\right\}, \quad (42)$$

where

$$\left[\begin{array}{c|c} A_\star & B_\star\\ \hline C_\star & 0\end{array}\right] \triangleq \left[\begin{array}{cc|c} A & B_2M_1 & B_2\\ 0 & A+B_2(M_1+C_{\rm c}) & 0\\ \hline C_2 & 0 & 0\\ 0 & 0 & 0\end{array}\right]. \quad (43)$$

Remark 4. Equation (42) has the unique stabilizing positive definite solution $T_\star = T_\star^{\rm T} \in \mathbb{R}^{2n\times 2n}$ such that the matrix $A_\star + B_\star N_\star$ is stable (Molinari (1975)).

Remark 5. Denote that for $U \equiv U^\circ$ and $V \equiv 0$

$$J_{\rm LQ}^\circ \triangleq \min J_{\rm LQ} = \mathbf{E}\left(\begin{bmatrix}\widehat{x}_0\\ \widehat{x}_0^0\end{bmatrix}^{\rm T} T_\star \begin{bmatrix}\widehat{x}_0\\ \widehat{x}_0^0\end{bmatrix}\right).$$

The following lemma defines the solution to Problems 1 and 2.

Lemma 4. Let the state-space realization matrices of stabilizing controller (11) be given by relations (36) of Lemma 3 together with

$$C_{\rm c} = N_1 + N_2, \quad (44)$$

where the matrices N_1 and N_2 are expressed via the stabilizing solution of control algebraic Riccati equation (42). Then controller (11) is a solution to Problems 1 and 2.

PROOF. Let us substitute (44) and (40) to (39) and apply Lemma 8 from Appendix. Taking into account (36) and (44), we obtain

$$K(z) \sim \left[\begin{array}{c|c} A + B_2(C_{\rm c}+M_1) - \Lambda(C_2+M_2) & \Lambda\\ \hline N_1+N_2 & 0\end{array}\right] = \left[\begin{array}{c|c} A_{\rm c} & B_{\rm c}\\ \hline C_{\rm c} & 0\end{array}\right].$$

Lemmas 2, 3, and 4 establish a system of matrix algebraic nonlinear equations for finding the state-space realization matrices of optimal controller (11) solving normalized anisotropy-based $\mathcal{H}_\infty$ problem (7) for n-dimensional plant (1). This system includes the following cross-coupled equations: $(2n \times 2n)$ Riccati equation (14) for the worst-case generating filter, $(2n \times 2n)$ Lyapunov equation (17), mean anisotropy equation (15), $(n \times n)$ filtering Riccati equation (34), $(2n \times 2n)$ control Riccati equation (42), expressions (36) and (44) for the controller matrices, as well as notational relations (12), (18), (19), (32), (37), and (43). Satisfying this equation system is sufficient for optimality of the obtained n-dimensional controller. Denote that in the case of zero mean anisotropy level $\alpha = 0$ the solution to Problem 1 reduces to the solution to normalized LQG problem considered by Jonkhere and Silverman (1983), Mustafa and Glover (1991), and the above equation system reduces to the well-known two independent $(n \times n)$ filtering and control Riccati equations that can be solved separately. But in general case $\alpha > 0$ the full cross-coupled equation system is solved numerically by means of specifically designed homotopy-based algorithm (see, for example, Kurdyukov et al. (2006)) with the normalized LQG controller state-space realization matrices as an initial point.

3. CONTROLLER ORDER REDUCTION BY ANISOTROPIC BALANCED TRUNCATION

3.1 *Anisotropic Characteristic Values and Anisotropic Balanced Coordinates*

To introduce a new set of invariants for the anisotropic optimal closed-loop system, let us consider filtering and control algebraic Riccati equations (34) and (42) with the respective stabilizing solutions S and $T_\star$.

Define the block partitioning

$$T_\star = \begin{bmatrix} T_{11} & T_{12}\\ T_{12}^{\rm T} & T_{22}\end{bmatrix}, \quad T_{ij} \in \mathbb{R}^{n\times n}, \quad (45)$$

of the stabilizing solution of control Riccati equation (42). Taking into account partitioning (45) and notation (43), equation (42) can be rewritten as

$$\left.\begin{aligned} T_{11} &= A^{\rm T}T_{11}A + C_2^{\rm T}C_2 - N_1^{\rm T}\Pi N_1\\ T_{12} &= A^{\rm T}T_{11}B_2M_1 + A^{\rm T}T_{12}(A+B_2M_1+B_2C_{\rm c})\\ &\quad -N_1^{\rm T}\Pi N_2\\ T_{22} &= (A+B_2M_1+B_2C_{\rm c})^{\rm T}T_{22}(A+B_2M_1+B_2C_{\rm c})\\ &\quad +(B_2M_1)^{\rm T}T_{11}(B_2M_1) - N_2^{\rm T}\Pi N_2\\ &\quad +(B_2M_1)^{\rm T}T_{12}(A+B_2M_1+B_2C_{\rm c})\\ &\quad +(A+B_2M_1+B_2C_{\rm c})^{\rm T}T_{12}^{\rm T}(B_2M_1)\\ \Pi &\triangleq B_2^{\rm T}T_{11}B_2 + I_{m_2}\\ N_1 &\triangleq -\Pi^{-1}B_2^{\rm T}T_{11}A\\ N_2 &\triangleq -\Pi^{-1}B_2^{\rm T}(T_{11}B_2M_1 + T_{12}(A+B_2M_1+B_2C_{\rm c}))\end{aligned}\right\} \quad (46)$$

with the matrix M_1 expressed by (37) through the stabilizing solution R of algebraic Riccati equation (14).

Remark 6. It has been noted by Tchaikovsky and Kurdyukov (2009) that if the state-space realization matrices of stabilizing controller (11) are given by relations (36) of Lemma 3 together with

$$\begin{aligned} C_c &= N_1 + N_2 \\ &= -(B_2^T(T_{11}+T_{12})B_2 + I_{m_2})^{-1} \\ &\quad \times B_2^T(T_{11}+T_{12})(A+B_2M_1), \end{aligned} \tag{47}$$

where the matrices $T_{11}, T_{12} \in \mathbb{R}^{n\times n}$ satisfy the first and second equations in system (46), then controller (11) is a solution to Problems 1 and 2.

In terms of block partitioning (45), for $U \equiv U^\circ$ and $V \equiv 0$ we have

$$\begin{aligned} J_{LQ}^\circ = \mathbf{E}(&\widehat{x}_0^T T_{11}\widehat{x}_0 + \widehat{x}_0^T T_{12}\widehat{x}_0^0 \\ &+(\widehat{x}_0^0)^T T_{12}^T \widehat{x}_0 + (\widehat{x}_0^0)^T T_{22}\widehat{x}_0^0). \end{aligned}$$

Since actually $\widehat{x}_0 \equiv \widehat{x}_0^0$ in the system closed by the state-estimating optimal controller, the above expression gives

$$\mathbf{E}(\widehat{x}_0^T T \widehat{x}_0) = \min J_{LQ} =$$
$$\min \mathbf{E}\left(\sum_{k=0}^{+\infty} \widehat{x}_k^T (C_2+N_1+N_2)^T (C_2+N_1+N_2)\widehat{x}_k\right), \tag{48}$$

where

$$T \triangleq T_{11} + T_{12} + T_{12}^T + T_{22}. \tag{49}$$

From (46), (47) and (49) it follows that the matrix $T = T^T > 0$ is the stabilizing solution of the following control Riccati equation

$$\left.\begin{aligned} T &= (A+B_2M_1)^T T(A+B_2M_1) \\ &\qquad + C_2^T C_2 - N^T \Pi N \\ \Pi &\triangleq B_2^T T B_2 + I_{m_2} \\ N &\triangleq -\Pi^{-1} B_2^T T(A+B_2M_1) \end{aligned}\right\}. \tag{50}$$

Since the matrix $A+B_2M_1$ is stable, equation (50) has the unique stabilizing solution (see Molinari (1975)).

Recall now that the stabilizing solution $S = S^T > 0$ of the filtering algebraic Riccati equation

$$\left.\begin{aligned} S &= (A+B_2L_{11})S(A+B_2L_{11})^T \\ &\qquad + B_2\Sigma_{11}B_2^T - \Lambda\Theta\Lambda^T \\ \Theta &\triangleq (C_2+L_{21})S(C_2+L_{21})^T + \Sigma_{22} \\ \Lambda &\triangleq ((A+B_2L_{11})S(C_2+L_{21})^T + B_2\Sigma_{12})\Theta^{-1} \end{aligned}\right\} \tag{51}$$

is the covariance matrix of the prediction error $\widetilde{x}_k = x_k - \widehat{x}_k$:

$$S = \mathbf{E}((x_k - \widehat{x}_k)(x_k - \widehat{x}_k)^T). \tag{52}$$

Now let us introduce a new set of invariants for the anisotropic optimal closed-loop system that will play a central role in reducing the order of the normalized anisotropic controller.

Theorem 5. Let the realization (A, B_2, C_2) of plant (1) be minimal and let $T = T^T > 0$ and $S = S^T > 0$ be the unique stabilizing solutions of control and filtering algebraic Riccati equations (50) and (51), respectively. Then the eigenvalues of the matrix TS are similarity invariants. Further, these eigenvalues are real and strictly positive. Let

$$\phi_1^2 \geqslant \phi_2^2 \geqslant \cdots \geqslant \phi_n^2 > 0$$

denote the n eigenvalues of the matrix TS arranged in decreasing order, then there exists a similarity transformation

$$(A, B_2, C_2) \to (Q^{-1}AQ, Q^{-1}B_2, C_2Q) \tag{53}$$

with the matrix Q nonsingular that transforms both T and S to the form

$$Q^{-1}TQ^{-T} = Q^T S Q = \Phi, \tag{54}$$

where

$$\Phi \triangleq \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_n \end{bmatrix}. \tag{55}$$

PROOF. Let us consider a nonsingular state transformation

$$x_k = Q\widetilde{x}_k, \quad h_k = Q\widetilde{h}_k, \quad Q \in \mathbb{R}^{n\times n}. \tag{56}$$

Substituting (56) to plant and controller state-space equations (1), (11), one can verify the correctness of (53). Further substitution of the realization $(Q^{-1}AQ, Q^{-1}B_2, C_2Q)$ to equations (14), (17), (15), (50), and (51), as well as to expressions (36) and (44) for the controller matrices and notational relations (12), (18), (19), (32), (37), and (43) shows that under transformation (56)

$$\begin{aligned} T &\to \widetilde{T} \triangleq Q^{-1}TQ^{-T}, \\ S &\to \widetilde{S} \triangleq Q^T S Q, \end{aligned}$$

that is

$$TS \to \widetilde{T}\widetilde{S} = Q^{-1}TSQ.$$

Since TS and $Q^{-1}TSQ$ are similar matrices, the eigenvalues of the matrix TS are similarity invariants. The existence of a similarity transformation resulting in equality (54) follows from the positive definiteness of T and S (see Corollary 8.3.3 in Bernstein (2005)), which also implies that the eigenvalues of the matrix TS are real and strictly positive.

Definition 1. The real positive values

$$\phi_1 \geqslant \phi_2 \geqslant \cdots \geqslant \phi_n > 0 \tag{57}$$

defined in Theorem 5 are called *anisotropic characteristic values.*

Definition 2. When the stabilizing solutions T and S of respective control and filtering Riccati equations (50) and (51) are in form (54), (55), the system is said to be in *anisotropic balanced coordinates*, and the realization

$$(\widetilde{A}, \widetilde{B}_2, \widetilde{C}_2) \triangleq (Q^{-1}AQ, Q^{-1}B_2, C_2Q)$$

is called *anisotropic balanced realization.*

Writing control and filtering algebraic Riccati equations (50) and (51) in the anisotropic balanced state-space representation $(\widetilde{A}, \widetilde{B}_2, \widetilde{C}_2)$ yields, respectively,

$$\left.\begin{aligned} \Phi &= (\widetilde{A} + \widetilde{B}_2\widetilde{M_1})^{\mathrm{T}}\Phi(\widetilde{A} + \widetilde{B}_2\widetilde{M_1}) \\ &\qquad + \widetilde{C}_2^{\mathrm{T}}\widetilde{C}_2 - \widetilde{N}^{\mathrm{T}}\widetilde{\Pi}\widetilde{N} \\ \widetilde{\Pi} &\triangleq \widetilde{B}_2^{\mathrm{T}}\Phi\widetilde{B}_2 + I_{m_2} \\ \widetilde{N} &\triangleq -\widetilde{\Pi}^{-1}\widetilde{B}_2^{\mathrm{T}}\Phi(\widetilde{A} + \widetilde{B}_2\widetilde{M_1}) \end{aligned}\right\}, \tag{58}$$

$$\left.\begin{aligned} \Phi &= (\widetilde{A} + \widetilde{B}_2\widetilde{L}_{11})\Phi(\widetilde{A} + \widetilde{B}_2\widetilde{L}_{11})^{\mathrm{T}} \\ &\qquad + \widetilde{B}_2\Sigma_{11}\widetilde{B}_2^{\mathrm{T}} - \widetilde{\Lambda}\widetilde{\Theta}\widetilde{\Lambda}^{\mathrm{T}} \\ \widetilde{\Theta} &\triangleq (\widetilde{C}_2 + \widetilde{L}_{21})\Phi(\widetilde{C}_2 + \widetilde{L}_{21})^{\mathrm{T}} + \Sigma_{22} \\ \widetilde{\Lambda} &\triangleq ((\widetilde{A} + \widetilde{B}_2\widetilde{L}_{11})\Phi(\widetilde{C}_2 + \widetilde{L}_{21})^{\mathrm{T}} \\ &\qquad + \widetilde{B}_2\Sigma_{12})\widetilde{\Theta}^{-1} \end{aligned}\right\} \tag{59}$$

with

$$\widetilde{M_1} \triangleq M_1Q, \quad \widetilde{L}_{11} \triangleq L_{11}Q, \quad \widetilde{L}_{21} \triangleq L_{21}Q. \tag{60}$$

The matrix Φ defined by (55) is the *unique* positive definite stabilizing solution to *both* of these algebraic Riccati equations, and due to this uniqueness all the relevant information related to the anisotropic characteristic values and the anisotropic balanced realization are concentrated in these two Riccati equations.

Remark 7. It must be understood that the anisotropic characteristic values just as the system anisotropic norm are functions of the external disturbance mean anisotropy level $\alpha \geqslant 0$. Strictly speaking, we should use the notation $\phi_i(\alpha)$ and $\Phi(\alpha)$, but for the sake of simplicity we apply notations ϕ_i and Φ.

Let $T_2 = T_2^{\mathrm{T}} > 0$ and $S_2 = S_2^{\mathrm{T}} > 0$ be the respective stabilizing solutions to dual control and filtering algebraic Riccati equations for the discrete-time LQG problem

$$\begin{aligned} T_2 &= A^{\mathrm{T}}T_2A + C_2^{\mathrm{T}}C_2 \\ &\quad - A^{\mathrm{T}}T_2B_2(B_2^{\mathrm{T}}T_2B_2 + I_{m_2})^{-1}B_2^{\mathrm{T}}T_2A, \end{aligned} \tag{61}$$

$$\begin{aligned} S_2 &= AS_2A^{\mathrm{T}} + B_2B_2^{\mathrm{T}} \\ &\quad - AS_2C_2^{\mathrm{T}}(C_2S_2C_2^{\mathrm{T}} + I_{m_2})^{-1}C_2S_2A^{\mathrm{T}}. \end{aligned} \tag{62}$$

Then one can define the LQG characteristic values for the discrete-time case similarly to Jonkhere and Silverman (1983) as

$$\begin{aligned} &\psi_1 \geqslant \psi_2 \geqslant \cdots \geqslant \psi_n > 0, \\ &\psi_i^2 = \lambda_i\{T_2S_2\}, \quad i = \overline{1, n}. \end{aligned} \tag{63}$$

The following theorem establishes some properties of the anisotropic characteristic values.

Theorem 6. Let the realization (A, B_2, C_2) of plant (1) be minimal, and let the LQG characteristic values for this realization be defined by (63). For anisotropic characteristic values (57), the following statements hold true:

(1) $\phi_i \geqslant \psi_i$ and $\phi_i = \psi_i$ iff $\alpha = 0$;
(2) each anisotropic characteristic value ϕ_i is a monotonically increasing function of the parameter α;
(3) if the anisotropic characteristic values ϕ_i are different, then $\frac{d\phi_i}{d\alpha} \geqslant 0$;
(4) each anisotropic characteristic value ϕ_i is a continuous function of the parameter α.

PROOF. (1) Applying the results of Wimmer (1992), Clements and Wimmer (1996) to equations (50), (61) and (34), (62), we obtain that

$$T \geqslant T_2 \quad \text{and} \quad S \geqslant S_2.$$

It follows that (see e.g. Bernstein (2005))

$$\begin{aligned} T^{1/2}ST^{1/2} &\geqslant T^{1/2}S_2T^{1/2}, \\ S_2^{1/2}TS_2^{1/2} &\geqslant S_2^{1/2}T_2S_2^{1/2} \end{aligned}$$

and

$$\lambda_i\{TS\} \geqslant \lambda_i\{TS_2\} \geqslant \lambda_i\{T_2S_2\}$$

which implies $\phi_i \geqslant \psi_i$. Equality is attained with $\alpha = 0$ since in this case $T = T_2$ and $S = S_2$ that completes the proof of the first assertion.

(2) It is known from Diamond et al. (2001) that the anisotropic norm of a system is a monotonically increasing differentiable function of the parameter α. It means that $\alpha_2 \geqslant \alpha_1 \geqslant 0$ always implies

$$\left\|\mathcal{F}_l(P, K(\alpha_2))\right\|_{\alpha_2} \geqslant \left\|\mathcal{F}_l(P, K(\alpha_2))\right\|_{\alpha_1}$$

with obvious notations. From the other hand, the anisotropic controller $K(\alpha_1)$ minimizes α_1-anisotropic norm of the closed-loop system that yields

$$\left\|\mathcal{F}_l(P, K(\alpha_2))\right\|_{\alpha_1} \geqslant \left\|\mathcal{F}_l(P, K(\alpha_1))\right\|_{\alpha_1}.$$

The resulting chain of inequalities

$$\begin{aligned} &\left\|\mathcal{F}_l(P, K(\alpha_2))\right\|_{\alpha_2} \\ &\qquad \geqslant \left\|\mathcal{F}_l(P, K(\alpha_2))\right\|_{\alpha_1} \\ &\qquad\qquad \geqslant \left\|\mathcal{F}_l(P, K(\alpha_1))\right\|_{\alpha_1} \end{aligned}$$

means that the anisotropic norm of the closed-loop system increases monotonically as a function of the parameter α. Applying again the results of Wimmer (1992), Clements and Wimmer (1996) to equations (50) and (34) obtained for the different mean anisotropy levels α_1 and α_2, it can be shown that

$$T(\alpha_2) \geqslant T(\alpha_1) \quad \text{and} \quad S(\alpha_2) \geqslant S(\alpha_1).$$

Using the same argument as in the proof of the first assertion, this implies that $\phi_i(\alpha_2) \geqslant \phi_i(\alpha_1)$.

In fact, T and S are differentiable functions of α, so we have

$$\frac{dT}{d\alpha} \geqslant 0 \quad \text{and} \quad \frac{dS}{d\alpha} \geqslant 0.$$

(3) Since T and S are differentiable functions of α, hence TS is also a differentiable function of α. Since by definition $\phi_i^2 = \lambda_i\{TS\}$ and by assumption ϕ_i are different, then each anisotropic characteristic value ϕ_i is a differentiable function of α too. But from the second assertion, each ϕ_i is a monotonically increasing function of α. Therefore, $\frac{d\phi_i}{d\alpha} \geqslant 0$ holds for each ϕ_i.

(4) From the proof of the third assertion, TS is a differentiable, hence, continuous function of α. It is well known (see e.g. Bernstein (2005)) that the eigenvalues of a matrix are continuous functions of the matrix elements that completes the proof.

Denote that the transformation matrix Q putting the closed-loop system realization into the anisotropic balanced coordinates can be found in the following quite standard way (see, for example, Datta (2004)). Let us find an upper triangular nonsingular matrix $\mathcal{T} \in \mathbb{R}^{n\times n}$ from the Cholesky factorization of the stabilizing solution T of control Riccati equation (50)

$$T = \mathcal{T}^{\mathrm{T}}\mathcal{T}$$

and the stabilizing solution S of filtering Riccati equation (51)

$$S = \mathcal{S}^{\mathrm{T}}\mathcal{S}.$$

Then, find the singular value decomposition of the matrix

$$\mathcal{S}\mathcal{T}^{\mathrm{T}} = \mathcal{U}\Phi\mathcal{V}^{\mathrm{T}},$$

where $\mathcal{U}\mathcal{U}^{\mathrm{T}} = I$, $\mathcal{V}\mathcal{V}^{\mathrm{T}} = I$. Then the transformation matrix is given by

$$Q = \mathcal{T}^{\mathrm{T}}\mathcal{V}\Phi^{-1/2}.$$

3.2 *Reduced-Order Plant and Controller*

Before proceeding to the controller order reduction, let us consider some motivation for the anisotropic balancing of Theorem 5 and summarize the essence of the results. First, the matrices $\widetilde{T}$ and $\widetilde{S}$ are diagonal. Therefore, taking into account expressions (48) for minimum value of the LQ-cost and (52) for the covariance matrix of prediction error, one could say that the similarity transformation Q defined by (53) decouples the state components in both the control and filtering problems. Second, since $\widetilde{T}$ and $\widetilde{S}$ are equal, this transformation also weights all of the state components equally between the control and the filtering problems. This weight or importance of the state component $\widetilde{X}_i$ in the normalized anisotropy-based stochastic $\mathcal{H}_\infty$ problem is the anisotropic characteristic value ϕ_i, since ϕ_i is the filtering error covariance for the component $\widetilde{X}_i$ and, at the same time, the cost induced by an initial condition aligned with $\widetilde{X}_i$. This fact carries over into the controller design problem, as the anisotropic optimal controller is the cascade of optimal estimator (35) and control gain (44). Thus, the anisotropic characteristic value ϕ_i specifies how much the state component $\widetilde{X}_i$ participates in the closed-loop behaviour of the system in the following sense. If ϕ_i is large, then the component $\widetilde{X}_i$ is difficult to filter (see (52) and (54)) and difficult to control (see (48) and (54)), hence, $\widetilde{X}_i$ is an important state component that must be taken into consideration in the controller design. Vice versa, if the anisotropic characteristic value is small, then $\widetilde{X}_i$ is easy to filter and easy to control, hence, the component $\widetilde{X}_i$ is not of the great essence that can be discarded for designing a reduced-order controller.

Let the state-space realization $(\widetilde{A}, \widetilde{B}_2, \widetilde{C}_2)$ be minimal with n states and in anisotropic balanced coordinates with anisotropic characteristic values

$$\phi_1 \geqslant \phi_2 \geqslant \cdots \geqslant \phi_n > 0.$$

That is, $\Phi = \operatorname{diag}\{\phi_1, \ldots, \phi_n\} = \widetilde{T} = \widetilde{S}$ is the stabilizing solution of control and filtering Riccati equations (58) and (59) associated with the realization $(\widetilde{A}, \widetilde{B}_2, \widetilde{C}_2)$. Fix $r < n$ such that $\phi_r > \phi_{r+1}$ and partition the matrix Φ accordingly into

$$\Phi = \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix} \tag{64}$$

with

$$\begin{aligned} \Phi_1 &= \operatorname{diag}\{\phi_1, \ldots, \phi_r\}, \\ \Phi_2 &= \operatorname{diag}\{\phi_{r+1}, \ldots, \phi_n\}. \end{aligned} \tag{65}$$

Partition the matrices $\widetilde{A}$, $\widetilde{B}_2$, and $\widetilde{C}_2$ conformably with the partitioning (64) as follows:

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \quad \widetilde{B}_2 = \begin{bmatrix} \widetilde{B}_{21} \\ \widetilde{B}_{22} \end{bmatrix},$$

$$\widetilde{C}_2 = \begin{bmatrix} \widetilde{C}_{21} & \widetilde{C}_{22} \end{bmatrix}.$$

Then the reduced-order realization with r-dimensional state is $(\widetilde{A}_{11}, \widetilde{B}_{21}, \widetilde{C}_{21})$.

Let $K(z) = (\widetilde{A}_{\mathrm{c}}, \widetilde{B}_{\mathrm{c}}, \widetilde{C}_c)$ be the normalized anisotropic controller for plant (1) as defined in Lemma 4. Partition the matrices $\widetilde{A}_{\mathrm{c}}$, $\widetilde{B}_{\mathrm{c}}$, and $\widetilde{C}_{\mathrm{c}}$ conformably with the partitioning (64) as follows:

$$\widetilde{A}_{\mathrm{c}} = \begin{bmatrix} \widetilde{A}_{\mathrm{c}11} & \widetilde{A}_{\mathrm{c}12} \\ \widetilde{A}_{\mathrm{c}21} & \widetilde{A}_{\mathrm{c}22} \end{bmatrix}, \quad \widetilde{B}_{\mathrm{c}} = \begin{bmatrix} \widetilde{B}_{\mathrm{c}1} \\ \widetilde{B}_{\mathrm{c}2} \end{bmatrix},$$

$$\widetilde{C}_{\mathrm{c}} = \begin{bmatrix} \widetilde{C}_{\mathrm{c}1} & \widetilde{C}_{\mathrm{c}2} \end{bmatrix}.$$

Then the reduced-order controller with r-dimensional state is

$$K_r(z) \sim \left[\begin{array}{c|c} \widetilde{A}_{c11} & \widetilde{B}_{c1} \\ \hline \widetilde{C}_{c1} & 0 \end{array}\right]. \tag{66}$$

As it can be easily shown by substitution, the matrix Φ_1 given by (65) is the stabilizing solution to the corresponding control and filtering Riccati equations

$$\left.\begin{array}{ll} \Phi_1 & = (\widetilde{A}_{11} + \widetilde{B}_{21}\widetilde{M}_{11})^{\mathrm{T}}\Phi_1(\widetilde{A}_{11} + \widetilde{B}_{21}\widetilde{M}_{11}) \\ & \qquad + \widetilde{C}_{21}^{\mathrm{T}}\widetilde{C}_{21} - \widetilde{N}_1^{\mathrm{T}}\widetilde{\Pi}_{11}\widetilde{N}_1 \\ \widetilde{\Pi}_{11} & \triangleq \widetilde{B}_{21}^{\mathrm{T}}\Phi_1\widetilde{B}_{21} + I_{m_2} \\ \widetilde{N}_1 & \triangleq -\widetilde{\Pi}_{11}^{-1}\widetilde{B}_{21}^{\mathrm{T}}\Phi_1(\widetilde{A}_{11} + \widetilde{B}_{21}\widetilde{M}_{11}) \end{array}\right\},$$

$$\left.\begin{array}{ll} \Phi_1 & = (\widetilde{A}_{11} + \widetilde{B}_{21}\widetilde{L}_{11})\Phi_1(\widetilde{A}_{11} + \widetilde{B}_{21}\widetilde{L}_{11})^{\mathrm{T}} \\ & \qquad + \widetilde{B}_{21}\Sigma_{11}\widetilde{B}_{21}^{\mathrm{T}} - \widetilde{\Lambda}_1\widetilde{\Theta}_{11}\widetilde{\Lambda}_1^{\mathrm{T}} \\ \widetilde{\Theta}_{11} & \triangleq (\widetilde{C}_{21} + \widetilde{L}_{21})\Phi_1(\widetilde{C}_{21} + \widetilde{L}_{21})^{\mathrm{T}} + \Sigma_{22} \\ \widetilde{\Lambda}_1 & \triangleq ((\widetilde{A}_{11} + \widetilde{B}_{21}\widetilde{L}_{11})\Phi_1(\widetilde{C}_{21} + \widetilde{L}_{21})^{\mathrm{T}} \\ & \qquad + \widetilde{B}_{21}\Sigma_{12})\widetilde{\Theta}_{11}^{-1} \end{array}\right\},$$

for the reduced-order realization $(\widetilde{A}_{11}, \widetilde{B}_{21}, \widetilde{C}_{21})$. From this fact it immediately follows that the reduced-order controller $K_r(z)$ is the full-order normalized anisotropic optimal controller for the reduced-order plant $P_r(z)$ with the realization $(\widetilde{A}_{11}, \widetilde{B}_{21}, \widetilde{C}_{21})$.

Of course, there are two important questions of stability and performance of the closed-loop system when reduced-order controller (66) is connected to full-order plant (1). Development of apriori conditions for closed-loop stability is now in progress. The lack of performance in terms of the anisotropic norm can be expressed as

$$|\!|\!| F_e |\!|\!|_\alpha = |\!|\!| F - F_r |\!|\!|_\alpha, \tag{67}$$

where

$$F_e(z) \sim \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{\mathrm{cl}} & 0 & B_{\mathrm{cl}} \\ 0 & A_{\mathrm{cl}r} & B_{\mathrm{cl}r} \\ \hline C_{\mathrm{cl}} & -C_{\mathrm{cl}r} & 0 \end{array}\right] \tag{68}$$

is the error model, F is the closed-loop system defined by (12),

$$F_r(z) \sim \left[\begin{array}{c|c} A_{\mathrm{cl}r} & B_{\mathrm{cl}r} \\ \hline C_{\mathrm{cl}r} & 0 \end{array}\right] = \left[\begin{array}{cc|cc} A & B_2C_{\mathrm{c}1} & B_2 & 0 \\ B_{\mathrm{c}1}C_2 & A_{\mathrm{c}11} & 0 & B_{\mathrm{c}1} \\ \hline C_2 & 0 & 0 & 0 \\ 0 & C_{\mathrm{c}1} & 0 & 0 \end{array}\right] \tag{69}$$

is the closed-loop system with a reduced-order controller.

The following theorem defines the value of performance error (67).

Theorem 7. Let the full-order closed-loop system $F(z)$ be given by (12) and let the reduced-order closed-loop system $F_r(z)$ represented by (69) be stable. Then the anisotropic norm of the error model $F_e(z)$ with realization (68) is given by

$$|\!|\!| F_e(z) |\!|\!|_\alpha = \left\{ \frac{1}{q_e}\left(1 - \frac{m_1}{\mathrm{tr}(L_eP_eL_e^{\mathrm{T}} + \Sigma_e)}\right)\right\}^{1/2},$$

where $q_e \in [0, \|F_e\|_\infty^{-2})$, L_e, $\Sigma_e = \Sigma_e^{\mathrm{T}} > 0$, and $P_e = P_e^{\mathrm{T}} > 0$ satisfy the equation system

$$\left.\begin{array}{ll} R_e & = A_e^{\mathrm{T}}R_eA_e + q_eC_e^{\mathrm{T}}C_e + L_e^{\mathrm{T}}\Sigma_e^{-1}L_e \\ \Sigma_e & = (I_{m_1} - B_e^{\mathrm{T}}R_eB_e)^{-1} \\ L_e & = \Sigma_eB_e^{\mathrm{T}}R_eA_e \end{array}\right\}, \tag{70}$$

$$-\frac{1}{2}\ln\det\left\{\frac{m_1\Sigma_e}{\mathrm{tr}(L_eP_eL_e^{\mathrm{T}} + \Sigma_e)}\right\} = \alpha, \tag{71}$$

$$P_e = (A_e + B_eL_e)P_e(A_e + B_eL_e)^{\mathrm{T}} + B_e\Sigma_eB_e^{\mathrm{T}}. \tag{72}$$

At that, the solution $(q_e, R_e = R_e^{\mathrm{T}} > 0, P_e)$ to equation system (70)–(72) is the unique one.

Proof of this theorem immediately follows from Theorem 2 in Vladimirov et al. (1996-1) applied to error model (68).

4. APPLICATION EXAMPLE: LONGITUDINAL FLIGHT CONTROL

As an application example, let us briefly consider the problem of longitudinal flight control aimed at wind disturbance attenuation for aircraft in landing approach along glidepath with prescribed relative slope angle in presence of coloured random noises by means of reduced-order anisotropic optimal controller. More detailed problem statement and aircraft model can be found in Kurdyukov et al. (2004). The obtained control law minimizes the influence of actuator and measurement noises, as well as wind disturbance on deviations of airspeed ΔV and altitude Δh from prescribed values (controlled variables). Deviation of generalized elevators $\Delta\vartheta_{cy}$ and throttle lever $\Delta\delta_t$ are considered as aircraft control.

The anisotropic, LQG, and $\mathcal{H}_\infty$ controllers were designed for aircraft TU-154 in landing approach along glidepath with fixed relative slope angle $\theta_0 = -2.7$ deg. Nonlinear equations describing an aircraft longitudinal motion (see Kurdyukov et al. (2004)) were linearized in the trajectory point with airspeed $V_0 = 71.375$ m/sec and altitude $h_0 = 600$ m. The resulted standard plant model (1) has order $n = 6$.

The full-order normalized anisotropic optimal controllers was found for two different prescribed levels of mean anisotropy of random disturbances $\alpha_1 \leqslant 0.01$ and $\alpha_2 \leqslant 0.6$; the suboptimal $\mathcal{H}_\infty$ controller was obtained for $\gamma = 2.6523$. Then, the closed-loop systems with anisotropic, LQG, and $\mathcal{H}_\infty$ controllers were put into the balanced coordinates via the respective nonsingular transformations. Anisotropic, LQG, and $\mathcal{H}_\infty$ characteristic values for this problems are given in descending order in Table 1. Denote that the $\mathcal{H}_\infty$ reduced-order controller does not provide stability of the closed-loop system for $r < 5$. The reduced anisotropic and LQG controllers retain the stable closed-loop up to $r = 3$.

Table 1. LQG, $\mathcal{H}_\infty$, and Anisotropic Characteristic Values

i	LQG	$\mathcal{H}_\infty$	Anisotropic:	
			$\alpha = 0.01$	$\alpha = 0.6$
1	2.5102	2.6369	6.5624	37.3321
2	0.8492	0.8925	1.3688	7.1399
3	0.5362	0.5611	0.7050	2.5277
4	0.0879	0.0900	0.0905	0.0993
5	0.0681	0.0693	0.0694	0.0712
6	0.0119	0.012430	0.012432	0.0127

The results of computer simulation for systems closed by the reduced-order controllers are presented in Fig. 3 through 9. The deterministic horizontal and vertical components of wind disturbance are presented by the model in the form of vortex ring considered by Kurdyukov et al. (2004) (see Fig. 3). The worst-case random coloured noises are presented in Fig. 4 and 7. The transients and control signals in the closed-loop systems with various controllers are given in Fig. 5, 6 and 8, 9. Black-coloured curves in the plots correspond to the closed-loop system with anisotropic controllers, blue and red colours — to the systems closed by LQG and $\mathcal{H}_\infty$ controllers, respectively.

The numerical comparison results are brought together in Tables 4 and 5. The comparison shows that the maximum absolute deviation of airspeed ΔV is lesser for system with the LQG controllers, whereas the maximum absolute deviation of the prescribed altitude Δh is lesser for system closed by the $\mathcal{H}_\infty$ controller with $r = 5$. The maximum absolute value of the control signal $\Delta\vartheta_{cy}$ for $r = 5$ was shown by the $\mathcal{H}_\infty$ controller, and for $r = 3$ by the anisotropic one, whereas the minimum absolute amplitude was demonstrated by LQG controller in both cases. As for the maximum absolute value of $\Delta\delta_t$, for $r = 5$ it was given by the anisotropic controller, and for $r = 3$ by the LQG one.

Table 2. Comparison of Reduced-Order Controllers, $\alpha = 0.01$, $r = 5$

Controller:		LQG	Anisotropic	$\mathcal{H}_\infty$
$\max\|\Delta V\|$,	m/sec	9.625	11.11	12.56
$\max\|\Delta h\|$,	m	60.94	45.61	29.15
$\max\|\Delta\vartheta_{cy}\|$,	deg	14.12	18.49	29.66
$\max\|\Delta\delta_t\|$,	deg	4.407	4.605	3.683

Table 3. Comparison of Reduced-Order Controllers, $\alpha = 0.01$, $r = 3$

Controller:		LQG	Anisotropic
$\max\|\Delta V\|$,	m/sec	9.53	11.87
$\max\|\Delta h\|$,	m	60.79	46.84
$\max\|\Delta\vartheta_{cy}\|$,	deg	16.85	21.91
$\max\|\Delta\delta_t\|$,	deg	8.689	9.108

From Fig. 5 and 8 it can be seen that the control signals generated by the anisotropic controller are more smooth than that of the $\mathcal{H}_\infty$ controller, whereas the latter, owing to its conservatism, strives to counteract each element of the noise random sequence, interpreting it as a deterministic signal. In practice, such the control would not likely be physically realizable or would require using of additional smoothing filters.

Table 4. Comparison of Reduced-Order Controllers, $\alpha = 0.6$, $r = 5$

Controller:		LQG	Anisotropic	$\mathcal{H}_\infty$
$\max\|\Delta V\|$,	m/sec	9.652	11.69	12.84
$\max\|\Delta h\|$,	m	63.39	40.96	29.64
$\max\|\Delta\vartheta_{cy}\|$,	deg	14.5	23.1	31.13
$\max\|\Delta\delta_t\|$,	deg	5.089	4.636	3.832

Table 5. Comparison of Reduced-Order Controllers, $\alpha = 0.6$, $r = 3$

Controller:		LQG	Anisotropic
$\max\|\Delta V\|$,	m/sec	9.509	12.74
$\max\|\Delta h\|$,	m	56.73	38.39
$\max\|\Delta\vartheta_{cy}\|$,	deg	16.01	24.73
$\max\|\Delta\delta_t\|$,	deg	8.227	6.981

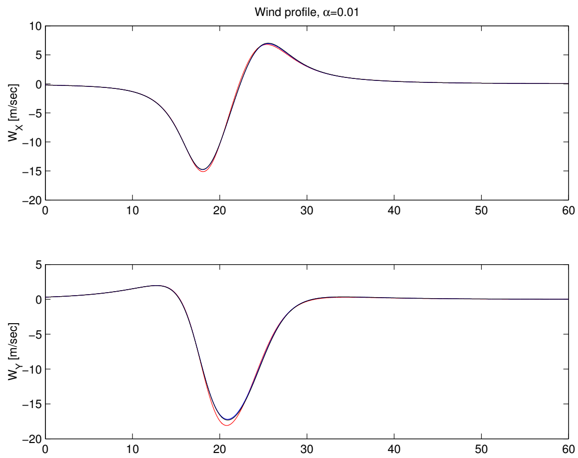

Fig. 3. Horizontal and vertical components W_x and W_y of wind profile

5. CONCLUSION

This paper presents the truncation technique for reducing order of normalized anisotropic optimal closed-loop system aimed at reduced-order controller design. Truncation is carried out for the closed-loop realization in anisotropic balanced coordinates, when the product of respective filtering and control Riccati equation solutions is a diagonal matrix with the squares of anisotropic characteristic values situated in descending order on its main diagonal. In anisotropic balanced coordinates, small characteristic values correspond to the states which are easy to filter and control in a sense of anisotropic norm. The part of the plant or controller corresponding to smaller anisotropic characteristic values is truncated to

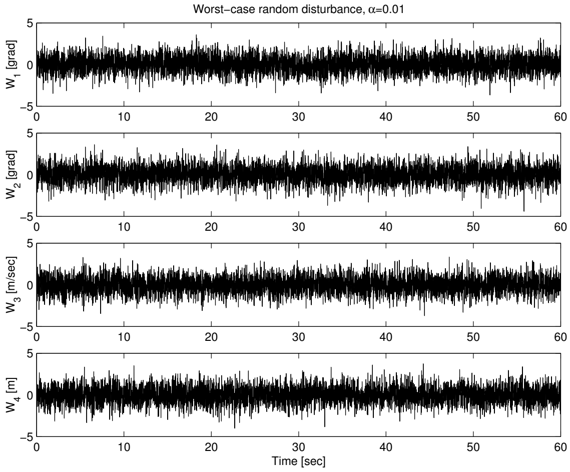

Fig. 4. Worst-case random disturbance, $\alpha = 0.01$

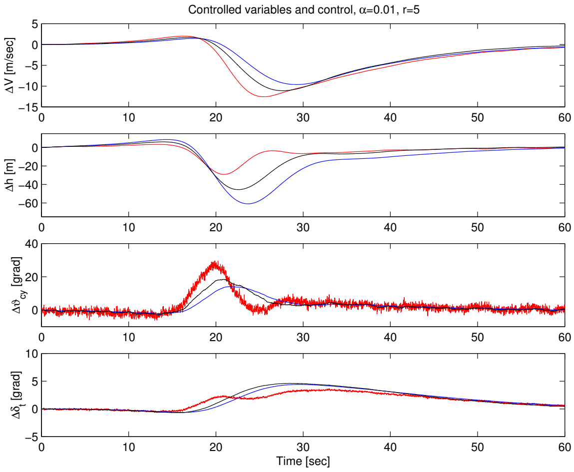

Fig. 5. Controlled variables ΔV, Δh and control $\Delta\vartheta_{cy}$, $\Delta\delta_t$, $\alpha = 0.01$, $r = 5$

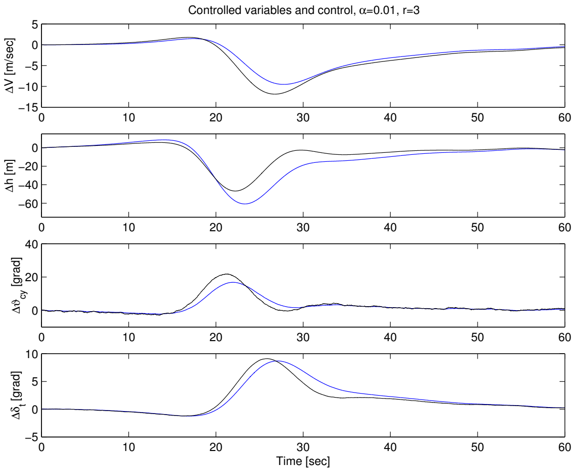

Fig. 6. Controlled variables ΔV, Δh and control $\Delta\vartheta_{cy}$, $\Delta\delta_t$, $\alpha = 0.01$, $r = 3$

obtain a reduced-order plant or controller. It was shown that the reduced-order controller is the full-order optimal one for the reduced-order plant. Development of apriori stability conditions for the closed-loop system consisting of full-order plant and reduced-order controller is now in progress.

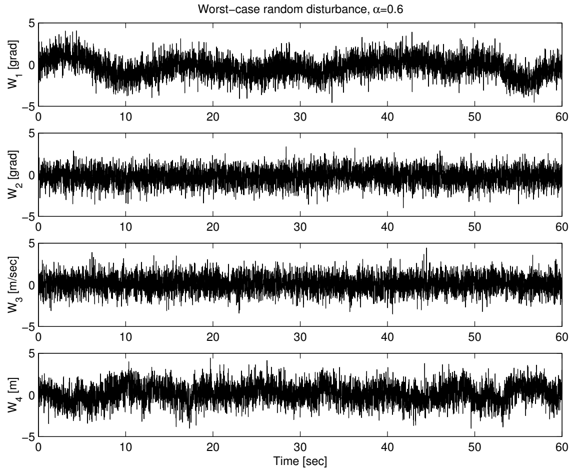

Fig. 7. Worst-case random disturbance, $\alpha = 0.6$

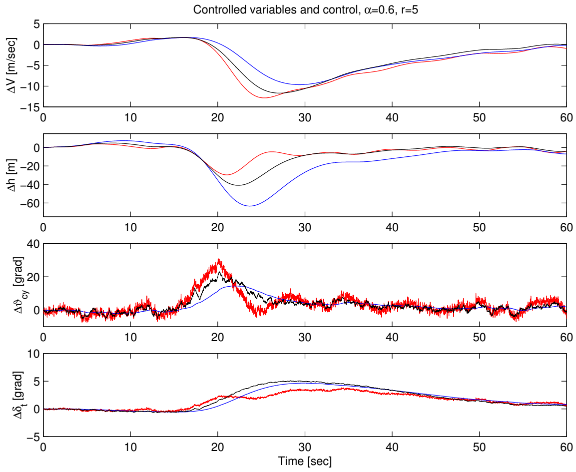

Fig. 8. Controlled variables ΔV, Δh and control $\Delta\vartheta_{cy}$, $\Delta\delta_t$, $\alpha = 0.6$, $r = 5$

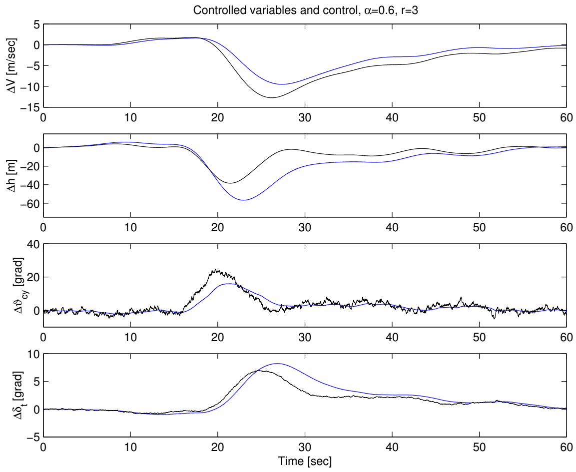

Fig. 9. Controlled variables ΔV, Δh and control $\Delta\vartheta_{cy}$, $\Delta\delta_t$, $\alpha = 0.6$, $r = 3$

As an application example, we consider the longitudinal aircraft control problem aimed at random disturbance attenuation by means of the reduced-order anisotropic controller. Simulation for aircraft in landing approach along glidepath with fixed relative slope angle shows that the reduced-order anisotropic controller retains the inherent properties of the full-order one. Comparison between reduced-order anisotropic, LQG, and $\mathcal{H}_\infty$ controllers in presence of the worst-case random and deterministic disturbances demonstrates the main advantages of the anisotropic controller, namely, smoothness and physical realizability of control signals together with sufficiently good attenuation of random and deterministic disturbances. In this problem, the reduced-order anisotropic controllers also show lesser order preserving closed-loop system stability than the reduced-order $\mathcal{H}_\infty$ controller.

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References

I.G. Vladimirov, A.P. Kurdjukov, and A.V. Semyonov. On Computing the Anisotropic Norm of Linear Discrete-Time-Invariant Systems. *Proc. 13th IFAC World Congress*, San-Francisco, CA, pages 179–184, 1996.

I.G. Vladimirov, A.P. Kurdjukov, and A.V. Semyonov. State-Space Solution to Anisotropy-Based Stochastic $\mathcal{H}_\infty$-Optimization Problem. *Proc. of 13th IFAC World Congress*, San-Francisco, CA, pages 427–432, 1996.

A.P. Kurdyukov and M.M. Tchaikovsky. Model Reduction According to Minimum Anisotropic Norm Performance. *Stability and Oscillations of Nonlinear Control Systems: Book of Abstracts of E.S. Pyatnitskiy X International Workshop*, Moscow, pages 166-167, 2008.

A.E. Jonckhere and L.M. Silverman. A New Set of Invariants for Linear Systems — Application to Reduced Order Compensator Design. *IEEE Trans. AC*, 28:953–964, 1983.

D. Mustafa and K. Glover. Controller Reduction by $\mathcal{H}_\infty$-Balanced Truncation. *IEEE Trans. AC*, 36:668–682, 1991.

R.S. Liptser and A.N. Shiryaev. *Statistics of Random Processes*. Springer-Verlag, New-York, 1977.

B.P. Molinari. The Stabilizing Solution of the Discrete Algebraic Riccati Equation. *IEEE Trans. AC*, AC-20:396–399, 1975.

P. Dorato and A.H. Levis. Optimal Linear Regulators: The Discrete-Time Case. *IEEE Trans. AC*, AC-16:613–620, 1971.

A.P. Kurdyukov, E.A. Maximov, and M.M. Tchaikovsky. Homotopy method for solving anisotropy-based stochastic $\mathcal{H}_\infty$-optimization problem with uncertainty. *Proc. 5th IFAC Symposium on Robust Control Design*, Toulouse, France, 2006.

M.M. Tchaikovsky and A.P. Kurdyukov. On Simplifying Solution to Normalized Anisotropy-Based Stochastic $\mathcal{H}_\infty$ Problem. *Appear in 6th IFAC Symposium on Robust Control Design*, Haifa, Israel, 2009.

D.S. Bernstein. *Matrix Mathematics: Theory, Facts, and Formulas with Application to Linear Systems Theory*. Princeton University Press, Princeton, NJ, 2005.

H.K. Wimmer. Monotonicity and Maximality of Solutions of Discrete-Time Algebraic Riccati Equations. *J. Math. Systems Estimat. Control*, 2:219–235, 1992.

D.J. Clements and H.K. Wimmer. Monotonicity of the Optimal Cost in the Discrete-Time Regulator Problem and Schur Complements. *Automatica*, 37:1779–1786, 2001.

P. Diamond, I.G. Vladimirov, A.P. Kurdyukov, and A.V. Semyonov. Anisotropy-Based Performance Analysis of Linear Discrete Time-Invariant Control Systems. *Int. J. of Control*, 74:28–42, 2001.

A.P. Kurdyukov, B.V. Pavlov, V.N. Timin, and I.G. Vladimirow. Longitudinal Anisotropy-Based Flight Control in a Wind Shear. *Proc. of 16th IFAC Symposium on Automatic Control in Aerospace*, Saint-Petersburg, Russia, 2004.

B.N. Datta. *Numerical Methods for Linear Control Systems Design and Analysis*. Elseveir Academic Press, San Diego, CA, 2004.

Appendix A. AN EQUIVALENCE LEMMA

Lemma 8. Let $A_1, A_2 \in \mathbb{R}^{n\times n}$. Then

$$\left[\begin{array}{cc|c} A_1 & A_2 - A_1 & B \\ 0 & A_2 & B \\ \hline C_1 & C_2 & D \end{array}\right] \sim \left[\begin{array}{c|c} A_2 & B \\ \hline C_1 + C_2 & D \end{array}\right], \qquad \text{(A.1)}$$

i.e. these realizations result in the same state-space input-output operator.

PROOF. Let $\left[\begin{smallmatrix} X \\ \Xi \end{smallmatrix}\right]$, U, and Y be the internal state, input, and output of the system with realization at the left-hand side of (A.1). All these signals are related by the equations

$$\left[\begin{array}{c} x_{k+1} \\ \xi_{k+1} \\ \hline y_k \end{array}\right] = \left[\begin{array}{cc|c} A_1 & A_2 - A_1 & B \\ 0 & A_2 & B \\ \hline C_1 & C_2 & D \end{array}\right] \left[\begin{array}{c} x_k \\ \xi_k \\ \hline u_k \end{array}\right]. \qquad \text{(A.2)}$$

Subtracting ξ_{k+1} from x_{k+1} in (A.2), we obtain

$$x_{k+1} - \xi_{k+1} = A_1(x_k - \xi_k)$$

that yields $x_k = \xi_k \ \forall k \in \mathbb{Z}$ for a coinciding initial conditions $x_{-\infty} = \xi_{-\infty}$, i.e. the second and third equations in (A.2) give the state-space realization at the right-hand side of (A.1).