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ROBUST CONTROL DESIGN FOR LTI SYSTEMS WITH REDUCED CONSERVATISM

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Abstract: A novel robust stability condition is proposed appropriate for the structured uncertain system matrix with a constant part; this is the case when a dynamic robust controller is designed (e.g. PID) and the augmented system includes both the uncertain controlled system and the fixed part respective to controller dynamics. Generally, the developed condition is less conservative than those from literature. Structured auxiliary matrices are used with a parameter dependent part respective to the fixed part of augmented system matrix, which reduces conservatism of the resulting robust stability condition. The resulting robust control design method applies BMI solver on the proposed stability condition. The properties of the method are illustrated on randomly generated examples.

Keywords: Robust control, Parameter dependent Lyapunov function, Polytopic uncertainty domain, Bilinear matrix inequalities

1 INTRODUCTION

Robustness belongs to important control design qualities when the results are to be implemented in practice. Modelling real plants inherently includes uncertainties (modelling errors due to linearization and approximation, disturbances etc.) that have to be considered in control design. Robust stabilization guarantees closed loop stability over the whole uncertainty domain. Recently, several approaches have been developed in robust control; the problem of robust stability and robust control is often formulated as an optimization problem. Recently, new efficient computational techniques have been developed to solve optimization tasks that can be formulated in form of Linear Matrix Inequalities (LMI). LMI techniques enable to solve a large set of convex problems in polynomial time (e.g. Boyd *et al.* 1994). Significant effort has been made to formulate various robust control problems in algebraic framework (Skelton *et al.* 1998) and transform them to LMI. This approach can be directly applied when control problems for linear systems with a convex (affine or polytopic) uncertainty domain are solved. However, there are many important control problems even for linear systems, which have been proved to be NP hard (Blondel and Tsitsiklis 1997). Generally, the class of structured linear control problems such as decentralized control and simultaneous SOF belong to NP hard problems. Nevertheless, various techniques have been developed to reformulate the problem as LMI one using certain convex relaxations as linearizing or convexifying functions (e.g. deOliveira *et al.* 2000; Rosinová and Veselý 2003). Another possible way is to formulate and solve the bilinear matrix inequalities (BMI) respective to robust control design problem. A nice review and basic characteristics of LMI and BMI in various control problems are in (Van Antwerp and Braatz 2000). Various algorithms to enhance numerical tractability of these nonconvex problems have been developed e.g. (Tuan *et al.* 2000).

In robust control of linear systems polytopic uncertainty domain is often considered since it has nice properties as a convex envelope of its vertices. For this uncertainty model, the notion of quadratic stability has been introduced considering one Lyapunov function for the whole uncertainty domain. This approach includes robustness against arbitrarily quick changes of system parameters within the uncertainty domain; however for slowly varying systems quadratic stability yields too conservative results. Therefore the robust stability notion and parameter dependent Lyapunov function have been introduced to reduce conservatism, e.g. (deOliveira *et al*., 1999, Henrion *et al* 2002).

To authors knowledge one of the best robust stability conditions has been proposed in (Peaucelle *et al.* 2000), see (Grman *et al.* 2003). To minimize conservatism of the robust stability condition, two auxiliary matrices are included. These auxiliary matrices appear in product with the system matrix varying over the uncertainty domain. The auxiliary matrices therefore have to be fixed for the whole uncertainty domain to keep linearity in uncertainty parameters. However, when a dynamic controller structure is considered, the augmented system matrix has a fixed part associate with controller dynamics (i.e. for PID case). To further reduce conservatism of the stability criterion, structured uncertainty of the controlled system matrix is employed and the auxiliary matrices are structured respectively. Then the part associate to controller dynamics is considered as parameter dependent (Duan *et al*. 2006; Gao *et al*. 2008). Another approach to reduce conservatism of stability criterion is using the polynomial parameter dependent Lyapunov function which, however, leads to enormously increasing number of unknown variables (Ebihara *et al* 2006).

In this paper we apply the approach described in previous paragraph and use structured auxiliary matrices to design a robustly stabilizing controller. The proposed use of structured matrices leads to "structured robust stability condition" which is the base for robust controller design. The resulting matrix inequality is solved as BMI. All developments are performed in general formulation including both discrete-time and continuous-time systems. The proposed robust control design approach has been verified on a PI robust controller design for randomly generated examples.

The structure of paper is following. In Section 2 the robust control problem is formulated for both continuous and discrete-time controllers. In Section 3 the novel robust stability condition employing structured auxiliary matrices is presented. The controller design algorithm based on BMI solution is illustrated in Section 4 on the randomly generated set of uncertain discrete-time systems.

2 PROBLEM FORMULATION AND PRELIMINARIES

Consider a linear affine uncertain system in general form covering both continuous and discrete-time cases:

$$\begin{aligned} \delta x(t) &= (A_0 + \delta A)x(t) + (B_0 + \delta B)u(t) \\ y(t) &= C_0 x(t) \end{aligned} \tag{1}$$

where

$\delta x(t) = \dot{x}(t)$ for continuous - time system

$\delta x(t) = x(t+1)$ for discrete - time system

$x(t) \in R^n, u(t) \in R^m, y(t) \in R^l$ are system state, control and output vectors respectively; A_0, B_0, C_0 are known constant matrices of appropriate dimensions corresponding to the nominal system; $\delta A, \delta B$ are matrices of uncertainties of the respective dimensions. In the following we consider square systems, i.e. m=l. For the affine uncertainties $\delta A, \delta B$ the respective equivalent polytopic model is used, given by its vertices

$$\{(A_1, B_1, C), (A_2, B_2, C), ..., (A_N, B_N, C)\} \tag{2}$$

where N is the respective number of polytope vertices.

Consider a controller with dynamics described by

$$\begin{aligned} \delta z(t) &= A_R z(t) + B_R e(t) \\ u(t) &= K_1 z(t) + K_2 e(t) \end{aligned} \tag{3}$$

where $e(t) \in R^l, e(t) = w(t) - y(t)$ is the control error (without loss of generality $w(t)$=0 is considered); A_R, B_R are controller dynamics matrices; K_1, K_2 are controller gains to be designed. Matrices of controller dynamics are determined by the controller structure, e.g. for discrete-time PI controller we have

$A_R = B_R = I_m$, $I_m \in R^{m \times m}$ is identity matrix.

Combining (1) and (3), the augmented closed loop system is obtained

$$\begin{bmatrix} \delta x(t) \\ \delta z(t) \end{bmatrix} = \begin{bmatrix} A_0 + \delta A & 0 \\ -B_R C & A_R \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} B_0 + \delta B \\ 0 \end{bmatrix} \begin{bmatrix} -K_2 & K_1 \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} \tag{4}$$

or alternatively, using the polytopic model (2) the closed loop dynamics is described as

$$\delta x(t) = A_C(\alpha) x(t) \tag{5}$$

where

$$A_C(\alpha) \in \left\{ \sum_{i=1}^{N} \alpha_i A_{Ci}, \quad \sum_{i=1}^{N} \alpha_i = 1, \quad \alpha_i \geq 0 \right\}$$

$$A_{ci} = \begin{bmatrix} A_i - B_i K_2 C & B_i K_1 \\ -B_R C & A_R \end{bmatrix} = \begin{bmatrix} A_{11i} & A_{12i} \\ A_{21} & A_{22} \end{bmatrix} \tag{6}$$

Note that while the upper blocks of the closed loop matrix A_{11i}, A_{12i} are parameter dependent (varying depending on their position in the uncertainty domain), the lower ones A_{21}, A_{22} are constant for whole uncertainty domain - they are associate with controller, which is fixed for the whole uncertainty domain.

This structure of system matrices – with lower part fixed for the whole uncertainty domain - enables to relax the stability condition used for robust controller design in Section 3.

The main aim of this paper is to design robust stabilizing controller for the uncertain system (1), in other words to find such control gain matrices K_1, K_2 which guarantee that the closed loop uncertain system (5),(6) is parameter dependent, quadratically stable within the whole uncertainty domain (2), (or (6) for closed loop system).

Recall briefly the basic notions concerning Lyapunov stability. To cover both continuous and discrete time cases, the D-stability concept (e.g. Henrion, 2002) has been used. In the following, * denotes transpose complex conjugate.

Definition 1 (D-stability)

Consider the D-domain in the complex plain defined as

$$D = \{s \text{ is complex number} : \begin{bmatrix} 1 \\ s \end{bmatrix}^* \begin{bmatrix} r_{11} & r_{12} \\ r_{12}^* & r_{22} \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0\}.$$

The considered linear system (1) is D-stable if all its poles lie inside the D-domain.

In the following, the standard choice of r_{ij} is used:

$r_{11} = 0, r_{12} = 1, r_{22} = 0$ (for a continuous-time system);

$r_{11} = -1, r_{12} = 0, r_{22} = 1$ (for a discrete-time system).

Quadratic D-stability is equivalent to the existence of one (parameter independent) Lyapunov function for the whole set describing the uncertain system model; this approach guarantees stability over the whole uncertainty domain for uncertain time invariant systems as well as for time variant systems where the parameters can vary at any rate within the uncertainty domain.

The robust stability notion based on the *parameter dependent* Lyapunov function (PDLF) defined as

$$P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i \quad \text{where } P_i = P_i^T > 0 \qquad (7)$$

α_i are respective to those in (6)

has been introduced to obtain less conservative results than those obtained using quadratic stability with a unique Lyapunov function.

Definition 2 (deOliveira *et al.* 1999)

System (5) is robustly D-stable in the convex uncertainty domain (6) with parameter-dependent Lyapunov function (7) if and only if there exists a matrix $P(\alpha) = P(\alpha)^T > 0$ such that

$$r_{12} P(\alpha) A_C(\alpha) + r_{12}^* A_C^T(\alpha) P(\alpha) + r_{11} P(\alpha) + \\ + r_{22} A_C^T(\alpha) P(\alpha) A_C(\alpha) < 0 \qquad (8)$$

for all α such that $A_C(\alpha)$ is given by (6).

Recall the sufficient robust D-stability condition proposed in (Peaucelle *et al*. 2000),

Lemma 1

If there exist matrices $E \in R^{nxn}, G \in R^{nxn}$ and N symmetric positive definite matrices $P_i \in R^{nxn}$ such that for all $i = 1, \ldots, N$:

$$\begin{bmatrix} r_{11}P_i + A_{Ci}^T E^T + E A_{Ci} & r_{12}P_i - E + A_{Ci}^T G^T \\ r_{12}^* P_i - E^T + G A_{Ci} & r_{22}P_i - (G + G^T) \end{bmatrix} < 0 \qquad (9)$$

then the system (5) is robustly D-stable.

Note that the auxiliary matrices E and G are constant for the whole uncertainty domain which makes (9) linear in i.

In the sequel, $X > 0$ denotes a positive definite matrix; * in matrices denotes the respective transposed term to make the matrix symmetric; I is an identity matrix and 0 is a zero matrix of the respective dimensions.

3 ROBUST STABILITY CONDITION FOR PARAMETER DEPENDENT AUXILIARY MATRICES

In this section the novel robust stability condition is developed from (9) employing the closed loop system matrix structure (6) to further decrease a certain conservatism still present in (9). This aim is reached by considering more general form of auxiliary matrices E and G in (9), with parameter dependent blocks corresponding to the constant part of the system matrix (6). In this way, known structure of the closed loop uncertainty enables to relax the stability condition (9) further used for the robust controller design.

The idea has been motivated by (Duan *et al.* 2006; Gao *et al*. 2008) where structured slack matrices have been used for the H_2 and H_{inf} filtering problems. The crucial point is to choose such a structure of the auxiliary matrices E and G that resulting products EA_{ci}, GA_{ci} do not destroy linearity in i of the respective stability condition, therefore to guarantee stability it is sufficient if this condition is satisfied in the vertices of the convex domain (6).

To achieve the outlined aim, the structure of E and G respective to the system matrix (6) is considered:

$$E_i = \begin{bmatrix} E_{11} & E_{12i} \\ E_{21} & E_{22i} \end{bmatrix}, G_i = \begin{bmatrix} G_{11} & G_{12i} \\ G_{21} & G_{22i} \end{bmatrix} \qquad (10)$$

where dimensions of the blocks of E_i, G_i correspond to the structure of A_{ci} defined in (6); index i denotes the uncertainty domain parameter dependence analogous to (7). The Lyapunov matrix (7) is structured as follows

$$P_i = \begin{bmatrix} P_{11i} & P_{12i} \\ P_{12i}^T & P_{22i} \end{bmatrix}. \quad (11)$$

Considering structure of the matrices A_{ci}, E_i, G_i, P_i the main result - relaxed stability condition is obtained.

Theorem 1

If there exist matrices $E_i \in R^{nxn}, G_i \in R^{nxn}$ with blocks defined by (10), and N symmetric positive definite matrices $P_i \in R^{nxn}$ defined by (11) such that:

$$M_i = \begin{bmatrix} m_{11i} & m_{12i} & m_{13i} & m_{14i} \\ * & m_{22i} & m_{23i} & m_{24i} \\ * & * & m_{33i} & m_{34i} \\ * & * & * & m_{44i} \end{bmatrix} < 0 \quad (12)$$

for all $i = 1,\dots, N$

where entries of matrix M_i are:

$$m_{11i} = r_{11}P_{11i} + sym(E_{11}A_{11i} + E_{12i}A_{21})$$

$$m_{12i} = r_{11}P_{12i} + E_{11}A_{12i} + E_{12i}A_{22} + A_{11i}^T E_{21}^T + A_{21}^T E_{22i}^T$$

$$m_{13i} = r_{12}P_{11i} - E_{11} + A_{11i}^T G_{11}^T + A_{21}^T G_{12i}^T$$

$$m_{14i} = r_{12}P_{12i} - E_{12i} + A_{11i}^T G_{21}^T + A_{21}^T G_{22i}^T$$

$$m_{22i} = r_{11}P_{22i} + sym(E_{21}A_{12i} + E_{22i}A_{22})$$

$$m_{23i} = r_{12}P_{12i} - E_{21} + A_{12i}^T G_{11}^T + A_{22}^T G_{12i}^T$$

$$m_{24i} = r_{12}P_{22i} - E_{22i} + A_{12i}^T G_{21}^T + A_{22}^T G_{22i}^T$$

$$m_{33i} = r_{22}P_{11i} - G_{11} - G_{11}^T$$

$$m_{34i} = r_{22}P_{12i} - G_{12i} - G_{21}^T$$

$$m_{44i} = r_{22}P_{22i} - G_{22i} - G_{22i}^T$$

then the system (5) is robustly D-stable.

To simplify reading recall that according to (6):

$$A_{11i} = A_i - B_i K_2 C$$

$$A_{12i} = B_i K_1$$

$$A_{21} = -B_R C$$

$$A_{22} = A_R$$

Symbol $sym(X)$ denotes $X + X^T$.

Proof

The proof of Theorem 1 is straightforward: substituting structured matrices (6), (10) and (11) into sufficient robust stability condition (9), the matrix inequality (12) is received.

Note that (12) is linear in i and it is LMI for stability analysis, i.e. considering unknown matrices E_i, G_i, P_i.

Corollary 1

If there exist matrices K_1, K_2 such that (12) holds for A_{ci} defined by (6), then a controller (3) robustly stabilizes the uncertain system (1) within the polytopic uncertainty domain given by vertices (2).

Corollary 1 suggests the way to design a robust controller. Since (12) is bilinear with respect to the unknown matrices K_1, K_2, P_i, E_i, G_i it can be solved either using some convex relaxation technique yielding a respective LMI formulation, or directly using some BMI solver. We adopt the latter way and propose the robust controller design procedure by solving (12) as BMI for unknown matrices E_i, G_i, P_i (simultaneously for all vertices). Recall, that E_i, G_i are auxiliary matrices free of any constraints; P_i are symmetric positive definite (these matrices define parameter dependent Lyapunov function (7)). The proposed robust controller design procedure has been verified on the set of randomly generated examples both for discrete-time and continuous-time systems. The results has been compared with those received using “unstructured” robust stability condition (9).

The robust stability condition (12) is more general than (9) and includes the “unstructured” variant (9) as its special case. Therefore from theoretical point of view the results of the proposed control design method should be at least as good as those received using (9).

4 EXAMPLES

The robust controller design method presented in Section 3 has been tested on randomly generated examples. Solution of BMI (12) proposed in the previous section (for unknown E_i, G_i, P_i, K_1, K_2) has been in each example compared with the solution of “unstructured” BMI (9) (for unknown E, G, P_i, K). In this section the results are summarized for discrete-time PI controller design.

Square systems have been considered with equal number of inputs and outputs. For each example the polytopic uncertainty domain is specified by its vertices. The PI controller is described by

$$u(k) = K_P e(k) + K_I \sum_{i=0}^{k} e(k) \tag{13}$$

The corresponding state-space model of PI controller (13) is in the form of (3):

$$\begin{aligned} z(k+1) &= A_R z(k) + B_R e(k) \\ u(k) &= K_1 z(k) + K_2 e(k) \end{aligned} \tag{14}$$

where $A_R = B_R = I_m$, $I_m \in R^{m \times m}$ is identity matrix, m is number of system inputs/outputs for control; $K_1 = K_I$, $K_2 = K_P + K_I$.

Example 1

Consider 50 randomly generated uncertain discrete-time systems (1), (2) with 4 states, 1 input/output and two uncertainty domain vertices subject to BMI (12) and for comparison to BMI (9) as well. In all cases the uncertain system is unstable in some points of the uncertainty domain.

Two qualities have been followed in each case: i) success of the design – if a feasible solution has been obtained yielding a robustly stabilizing controller; ii) the maximum spectral radius of the respective uncertainty domain vertices. The results are summarized in Tab.1.

Method	Number of successful solutions	Average spectral radius
Novel BMI (12)	44	0.8139
BMI (9)	44	0.8142

Tab.1.

Example 2

Again, the 50 randomly generated discrete-time examples with 4 states and two vertices uncertainty domain, but with 2 inputs/outputs are considered. Each uncontrolled model is unstable in some points of the uncertainty domain. The results are summarized in Tab.1b.

Method	Number of successful solutions	Average spectral radius
Novel BMI (12)	41	0.8332
BMI (9)	40	0.8363

Tab.2

The results in both examples are very close to each other, though in the latter case there was one example where the new method outperforms the previous one. It can be expected that for more complex systems the results would favor the new method. The reason is that for more complex system with more inputs and outputs, the fixed part of the augmented system matrix respective to controller dynamics is bigger; therefore the relaxation brought about by structured auxiliary matrices would be also bigger. Unfortunately, the BMI solver we have used (PENBMI) has numerical problems with bigger systems (and more unknown variables), therefore at present we search other ways to prove qualities of our proposed stability condition in robust control design.

5 CONCLUSION

A new robust stability condition has been proposed appropriate for dynamic robust controller design (e.g. PID). The developed condition is generally less conservative than the ones known from literature. This quality comes from using structured auxiliary matrices with a parameter dependent part, which reduces conservatism of the robust stability condition. For robust control design, the proposed condition is in form of BMI. The BMI solver has been used to test the proposed robust control design method. The results obtained from randomly generated examples illustrate the benefits of the proposed method. Further possibilities of the proposed approach are under research.

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