On Inequality Path Constraints in Dynamic Optimisation

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Contents

Chapter 1

Introduction and Problem Definition

A general problem of dynamic optimisation includes definitions of a system to be optimised, cost function, and constraints on the manipulated and controlled variables. Among the constraints, inequality path constraints involving state variables are particularly difficult to handle. The purpose of this report is to make an overview of the existing methods capable to deal with these constraints and evaluate their performances.

The dynamic optimisation problem treated in the report is considered of the form

$$\min_{\boldsymbol{u},t_j} J \quad = \quad G(t_j, \boldsymbol{x}(t_f), \boldsymbol{p}) + \int_0^{t_f} F_0(t, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})\mathrm{d}t, \quad j = 1, \ldots, P \tag{1.1}$$

subject to system constraint:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{1.2}$$

and subject to inequality path constraints

$$\boldsymbol{0} \leq \boldsymbol{g}(t, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \tag{1.3}$$

where t denotes time, $\boldsymbol{x} \in \mathcal{R}_{n_x}$ is the vector of differential state variables, $\boldsymbol{u} \in \mathcal{R}_{n_u}$ is the vector of controls, and $\boldsymbol{p} \in \mathcal{R}_{n_p}$ is the vector of parameters. The vector valued functions $\boldsymbol{f} \in \mathcal{R}_{n_x}, \boldsymbol{g} \in \mathcal{R}_m$ describe right hand sides of differential equations and the inequality constraints, respectively.

The optimised variables are the control trajectory $\boldsymbol{u}$ parametrised by piece-wise constant trajectory and the commutation times (times when the piece-wise control changes) denoted by t_j, $j = 1, \ldots, P$. P is the number of piece-wise constant segments.

Chapter 2

Methods for Handling Inequality Path Constraints

There are some reviews published that discuss the issue of inequality path constraints (IPC) (Vassiliadis et al., 1994; Feehery and Barton, 1998).

In the discussion presented below we have not mentioned the use of stochastic techniques due to the fact that they are quite ineffective in the neighbourhood of the solution and require a huge number of the cost function evaluations. Similarly, the penalty function approach is not considered as it is closely related to the treatment via endpoint constraints.

2.1 Total Discretisation

The orthogonal collocation approach parametrises all states as low degree polynomials on finite time elements. The approximations used most frequently are the Lagrange polynomials.

The IPC are then transformed into a series of interior state inequality constraints treated directly within the nonlinear programming (NLP) algorithm. The disadvantage of this approach is large dimension of the resulting NLP problem.

2.2 Endpoint Constraints

An IPC can be transformed into an end-point constraint of the form

$$J_1 = \int_0^{t_f} h(t, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \mathrm{d}t = 0 \tag{2.1}$$

where h measures the degree of violation of the inequality constraint during the entire trajectory. Due to the convergence reasons of the NLP, the equality is often relaxed to

$$J_1 = \int_0^{t_f} h(t, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \mathrm{d}t < \varepsilon \tag{2.2}$$

Note that this means that a small violation of the constraint is allowed.

Several formulations for selection of a suitable h have been proposed:

Max The most simple approach is to use $h = \min(g_i, 0)$. As the gradient of this operator is discontinuous, it poses problems in the integration and is in general not recommended.

Max2 An improvement over the previous solution is to avoid the discontinuity, for example as with $h = [\min(g_i, 0)]^2$.

Smoothing The proposition given in Teo et al. (1991) lies in the discontinuity replacement by a smooth approximation. Therefore, in the region of $|g_i| < \delta$, a quadratic smoothing is employed:

$$h = \begin{cases} g_i & \text{if } g_i < -\delta \\ -\dfrac{(g_i - \delta)^2}{4\delta} & \text{if } |g_i| \leq \delta \\ 0 & \text{if } g_i > \delta \end{cases} \tag{2.3}$$

The advantage over the Max2 method is elimination of squaring, therefore small deviations are penalised more heavily. A drawback (in practice not very important) is slightly smaller feasibility region.

The disadvantage of these methods is that no distinction is made whether the path constraint is not active on the overall trajectory or it is exactly at the limit. In both cases the integral is zero. Moreover, if the path constraint is not violated, its gradient with respect to the optimised parameters is zero. This poses problems to NLP and reduces its convergence speed considerably as it oscillates between zero and nonzero gradients. Further, the gradients are zero at the optimum.

2.3 Interior point constraints

As the previous methods transformed the whole IPC into a single integral measure, only a very little information is given to NLP problem. A possible solution is to discretise the constraint and require it to be satisfied only at some points – usually at the end of the segments at times t_i. Thus,

$$g_i(t_j) \geq 0 \tag{2.4}$$

Although such a formulation is more easily solved by the NLP solver, it can result in pathological behaviour when the constraint is respected only at the prescribed points but violated in between. The situation gets worse if also the times t_i are optimised. The usual solution found contains one very large time t_k within which the constraint exceeds largely the possible tolerated violation.

The situation is better if the times are not optimised and tolerable constraint violations can be obtained. This approach is used in Model Predictive Control.

2.4 Combination of End and Interior Point Methods

A straightforward improvement consists in the combination of the integral end-point function described in the Section 2.2 with the set of interior point constraints at the end time segments from the Section 2.3. The advantage lies in the fact that the NLP solver gets more information even when the integral is zero.

2.5 Slack Variables

The slack variable method (Jacobson and Lele, 1969; Bryson and Ho, 1975; Feehery and Barton, 1998) is based on the techniques of optimal control and among other methods mentioned above it is one of the most rigorous. However, it has many drawbacks so it cannot be used as a general purpose method.

The principle of the method is given by changing the original inequality to equality by means of a slack variable $a(t)$:

$$g_i(x) - \frac{1}{2}a^2 = 0 \tag{2.5}$$

The slack variable is squared so that any value of a is admissible.

The equation is then differentiated so many times until a explicit solution for u can be found. The control variable is then eliminated from the system equations. Therefore, the principle of the method is to make the control (one scalar element from the control vector) a new state variable for the corresponding constraint. Thus, the state constraint is appended as equality to the system, the control variable is solved as a state variable during the integration and the slack variable $a(t)$ becomes the new optimised variable.

The original proposition in Jacobson and Lele (1969) dealt only with ODE (ordinary differential equations). Improvement and generalisation of this procedure for general DAE (differential algebraic systems) has been proposed by Feehery and Barton (1998).

Although the method is appealing and gives very good results, its drawbacks are as follows:

- Number of the state constraints cannot be larger than the dimension of the control vector. The other approaches can give a feasible solution when the number of *active* constraints is equal or smaller than the control dimension. A possible workaround has been proposed by Feehery and Barton (1998) where state event based approach is used.
- As the method changes the category of $\boldsymbol{u}$ from the optimised variable to a state variable, this means that any possible bounds on $\boldsymbol{u}$ cannot longer be satisfied. Therefore, a control constraint is traded versus a state constraint. As in the real world any control signal has bounds well defined, it remains only to hope that these will not be violated.
- If more control variables are candidates for state variables, no suitable selection strategy exists. In practice, usually a combination of control variables (not only one of them) has influence on the state constraint; the method is not capable to find it.
- It is implicitly given that the control variable can vary continuously in time. There are some optimisation problems where control has to be piece-wise constant – for example bang-bang type of control strategy where only time lengths can be varied.

On the other hand, the primary advantages are as follows

- Only feasible solutions are generated by the integration (IVP – initial value problem) solver. This may particularly be important when the solutions that violate the path constraint would generate feasibility problems to IVP solver.
- The constraints are respected within the integration precision of the IVP solver and need not to be relaxed for improved convergence of the optimisation.

2.6 Hybrid Method

As an intermediate approach between the total discretisation based on orthogonal collocation and other methods, a method based on a partial discretisation of selected state variables can be proposed. Assume for simplicity that the constraint $g_i(\boldsymbol{x})$ is a function of one state x_k only. Parametrising x_k as a (possibly the Lagrange) polynomial transforms its differential equation into a set of equality constraints. Furthermore, continuity constraint on this state has to be ensured between the time segments. The original state path constraint $g_i(x_k)$ is also transformed into a set of inequality constraints in optimised variables.

Compared to the total discretisation, smaller number of optimised variables in NLP results as the remaining differential equations are still solved by the integration.

Chapter 3

Comparison of Selected Methods for a Small Example

Let us consider the following dynamic optimisation problem treating LQ control of a linear time-invariant system (Jacobson and Lele, 1969; Logsdon and Biegler, 1989; Feehery, 1998):

$$\min_{u(t)} J = \int_0^1 (x_1^2 + x_2^2 + 0.005u^2)\mathrm{d}t \tag{3.1}$$

subject to:

$$\begin{aligned} \dot{x}_1 &= x_2, \quad x_1(0) = 0 & (3.2)\\ \dot{x}_2 &= -x_2 + u, \quad x_2(0) = -1 & (3.3)\\ 0 &\geq x_2 - 8(t-0.5)^2 + 0.5 & (3.4) \end{aligned}$$

Assume division of the time interval $[0, 1]$ to 10 finite elements of variable length and control parametrisation with one or two optimised parameters per finite element.

The problem can be rewritten with the methods described in the previous chapter as follows ($g = x_2 - 8(t - 0.5)^2 + 0.5$):

Max2 constraint:

$$\varepsilon - \int_0^1 (\max(g, 0))^2 \mathrm{d}t \geq 0 \tag{3.5}$$

Smoothing constraint:

$$\varepsilon - \int_0^1 h(t)\mathrm{d}t \geq 0 \tag{3.6}$$

where

$$h = \begin{cases} g & \text{if } g_i < -\delta \\ -\dfrac{(g-\delta)^2}{4\delta} & \text{if } |g| \leq \delta \\ 0 & \text{if } g > \delta \end{cases} \tag{3.7}$$

Points constraint:

$$x_2(t_i) - 8(t_i - 0.5)^2 + 0.5 \leq 0, \quad i = 1 \ldots 10 \tag{3.8}$$

where t_i are the ends of the finite elements.

Slack As the constraint is not a function of u, it is differentiated with respect to time, thus

$$x_2 - 8(t-0.5)^2 + 0.5 + \frac{1}{2}a^2 = 0 \quad (3.9)$$

$$\dot{x}_2 - 16(t-0.5) + a\dot{a} = 0, \quad a(0) = \sqrt{5} \quad (3.10)$$

The initial condition $a(0)$ follows from the constraint (3.4) at time $t = 0$ when both states are known.

Comparing (3.4) and (3.10) follows for u

$$u = x_2 + 16(t-0.5) - a\dot{a} \quad (3.11)$$

Now setting $a_1 = \dot{a}$ as an optimised variable (and constraining a not to be discontinuous) leads to an optimisation problem:

$$\min_{a_1(t)} J = \int_0^1 (x_1^2 + x_2^2 + 0.005(x_2 + 16(t-0.5) - aa_1)^2)\mathrm{d}t \quad (3.12)$$

subject to:

$$\dot{x}_1 = x_2, \quad x_1(0) = 0 \quad (3.13)$$

$$\dot{x}_2 = 16(t-0.5) - aa_1, \quad x_2(0) = -1 \quad (3.14)$$

$$\dot{a} = a_1, \quad a(0) = \sqrt{5} \quad (3.15)$$

and a_1 is parametrised as an optimised variable.

Is is also possible to use the DAE problem description

$$\min_{a(t)} J = \int_0^1 (x_1^2 + x_2^2 + 0.005u^2)\mathrm{d}t \quad (3.16)$$

subject to:

$$\dot{x}_1 = x_2, \quad x_1(0) = 0 \quad (3.17)$$

$$\dot{x}_2 = -x_2 + u, \quad x_2(0) = -1 \quad (3.18)$$

$$x_2 = -8(t-0.5)^2 + 0.5 + \frac{1}{2}a^2, \quad a(0) = \sqrt{5} \quad (3.19)$$

This DAE has differential index (Brenan et al., 1989) higher than one and has to be reduced to an index one formulation. Moreover, the optimised variable a has to be piecewise continuous, otherwise the state x_2 would exhibit discontinuities. This approach has not been further studied.

3.1 Results

The problem has been studied with the control parametrisation of the type $u(t) = u_1 + u_2 t$ (piece-wise linear with possible discontinuities) and with precisions of the IVP/NLP solvers either $10^{-7}/10^{-5}$ or $10^{-9}/10^{-7}$. Dynamic optimisation solver DYNO (Fikar, 2001) has been used. Statistics of the results are summarised in Table 3.1 for the slack variable approach (where also constant and quadratic control parametrisation has been used) and in Table 3.2 for other standard methods. No comparison of CPU/execution time has been performed as that is machine dependent. Instead, number of iterations is shown in the tables.

As the results in Table 3.1 indicate, the slack variable approach is quite insensitive to the choice of the working precision. The number of iterations and the attained cost function value remain

Table 3.1: Slack variable statistics

	J	Precision	Iterations (IVP, NLP)
Constant	0.1744	$10^{-7}/10^{-5}$	75/59
Constant	0.1740	$10^{-9}/10^{-7}$	123/105
Linear	0.1729	$10^{-7}/10^{-5}$	53/48
Linear	0.1729	$10^{-9}/10^{-7}$	73/68
Quadratic	0.1714	$10^{-9}/10^{-7}$	193/174

Table 3.2: Statistics for other standard methods

	J	Precision	Iterations (IVP, NLP)
Max2	0.1743	$10^{-7}/10^{-5}$	72/59
Max2	0.1677	$10^{-9}/10^{-7}$	1381/627
Max2+Point	0.1737	$10^{-7}/10^{-5}$	107/84
Max2+Point	0.1680	$10^{-9}/10^{-7}$	1306/656
Smoothing	0.1758	$10^{-7}/10^{-5}$	187/142
Smoothing	0.1703	$10^{-9}/10^{-7}$	680/490
Point (var. t_i)	0.0690	$10^{-7}/10^{-5}$	132/130
Point (fix. t_i)	0.2991	$10^{-7}/10^{-5}$	62/62

approximately constant for a given parametrisation. However, the type of the parametrisation is significant for the value of the cost, at least if only low degree polynomials are employed.

Figure 3.1 shows the comparison between the approximations for the control variable calculated from (3.11) and for the inequality constraint (3.4). It has to be mentioned that the constraint is indeed never violated. As the control u is allowed to vary, a flat maximum for a large time interval can be observed for the constraint. On the other hand, approximation of the slack cannot clearly be optimal and the cost function value is larger than in other approaches.

As mentioned before, it is impossible with this method to impose for example upper bound $u < 10$ or to add some other inequality constraint.

The methods (Max2, Max2+Point, Smoothing) share some common features (see Table 3.2):

- The (local) optimum found is strongly dependent on the working precision.
- The time needed to improve the number of significant digits in the cost function by setting more stringent precisions increases geometrically.
- There are no significant differences in performances of these methods either in precision, or in the number of iterations. Even if the smoothing method at the higher precision needs smaller number of iterations to come to the optimum, it is because the value of the cost function attained is higher in this case. The rate of convergence is about the same.

  Actually, the most simple approach Max2 seems to be quite satisfactory for this problem, even if the other two are claimed to be more efficient in the literature (Teo et al., 1991; Vassiliadis et al., 1994).

Figure 3.2 shows the actual differences between the slack method (quadratic approximation) and Max2 (precisions $10^{-9}/10^{-7}$). During the time when the constraint is active, the actual constraint trajectory is not exactly zero but oscillates around zero for the Max2 case. This is a characteristic feature of the integral methods, as the control is constrainted by some low level polynomial and the constraint is handled only by the NLP solver. The control trajectory in the slack method does not suffer from this problem as it is calculated by the nonlinear IVP solver during the active constraint and the constraint is satisfied within the integration precision. The

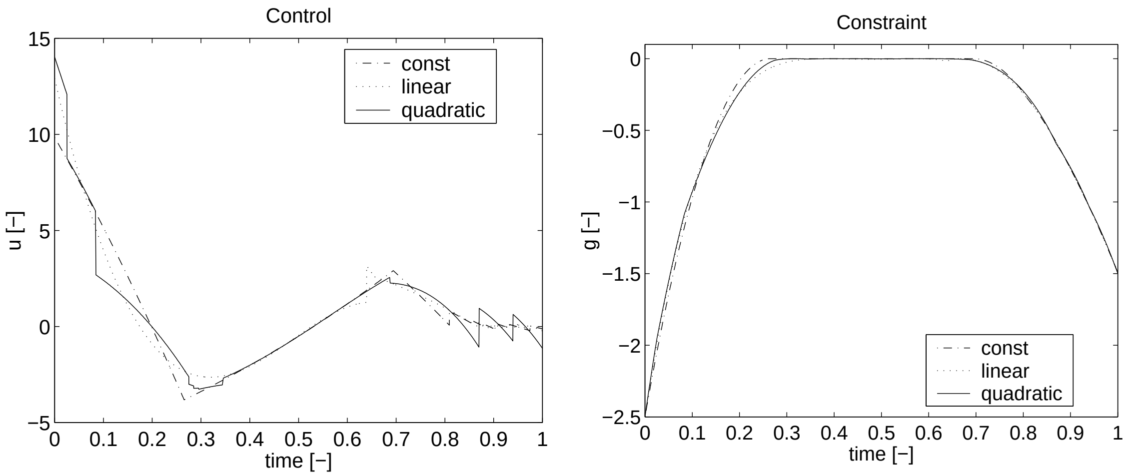

Figure 3.1: Comparison of different parametrisations (constant, linear, and quadratic) in the Slack approach. Left: control variable, right: state constraint

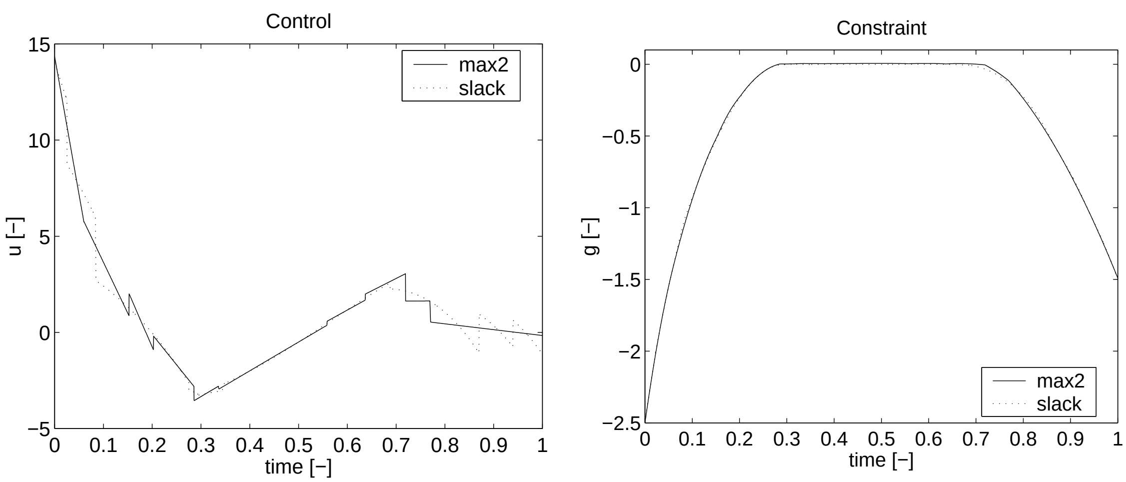

Figure 3.2: Comparison of Max2 and Slack approaches. Left: control variable, right: state constraint

cost function is less in the Max2 case; this is also caused by the fact that the integral constraint is allowed to be violated slightly by the amount $\varepsilon = 10^{-5}$.

Finally, the danger of using interior point constraints only is shown in Figure 3.3 and in the last two rows of Table 3.2. In the first case with variable finite time elements, the cost function is very small compared to other methods. What actually happens can be seen in Figure 3.3 (solid line right). One time element grows very large at optimum and even if the point constraints are well satisfied at both its ends (times $t_1 = 0.25, t_2 = 0.75$), the interval length allows the costraints to be very large in the middle.

Even though, the usual practice in predictive control is to employ this kind of constraint handling. However, as the finite time elements (sampling time) usually do not vary, this practice has some justification as the dotted line in Figure 3.3 (right) shows. The maximum of the constraint is about 0.07 which may often be quite acceptable. The drawback however, is the value of the cost function (Table 3.2, last row) that is almost twice of the best minimum value.

Some other approaches have been tested in order to find some improvement for the case of integral (Max2) approximations:

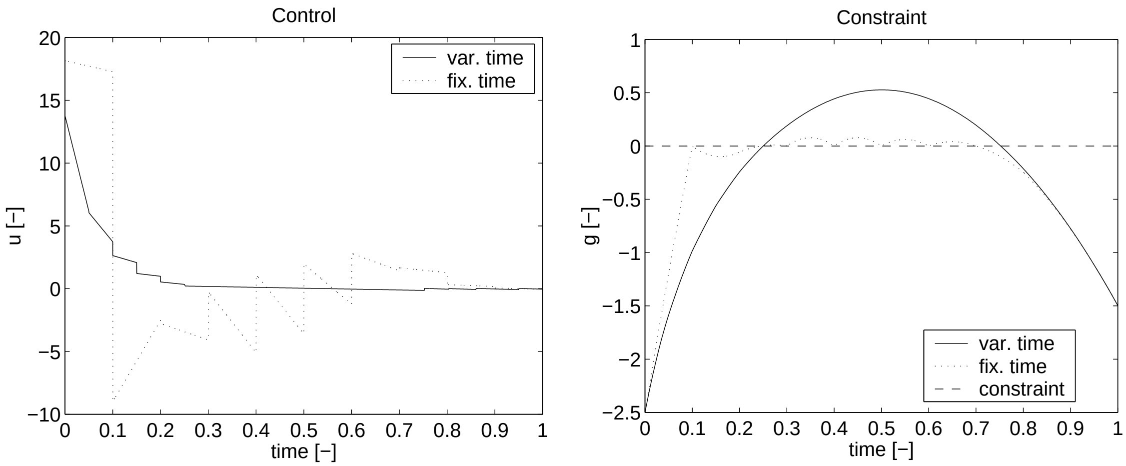

Figure 3.3: Interior point constraints only with either fixed or variable finite time lenghts. Left: control variable, right: state constraint

- One of the issues of the Max2 method lies in the fact that no distinction is made whether the path constraint is not active on the overall trajectory or it is exactly at the limit and in both cases the integral is zero.

  A possible improvement is to enable from the beginning a possible violation of the constraint with the modified h as

$$h = \begin{cases} g/k & \text{if } g < -\delta \\ \bar{h}(\delta) & \text{if } |g_i| \leq \delta, \qquad k \gg 1 \\ gk & \text{if } g_i > \delta \end{cases} \tag{3.20}$$

  and $\bar{h} = a + bg + cg^2 + dg^3$ is a third order polynomial fitted so that its value and the value of its derivative are equal at the both ends to the values of the corresponding lines.

  This choice results in the integral that it grows rapidly in the negative direction if the constraint is violated. It is positive if only a small violations can be observed. The principle is to penalise more heavily the undesired phenomena. Moreover, the constraint which is well defined does not contain hidden discontinuities and is integrated by the IVP solver more easily.

  A possible drawback is that k acts as a penalty function and thus shares its inconveniences. As k grows, the problem becomes more nonlinear and NLP converges more slowly.

- Another problem in Max2 method is that the whole constraint trajectory is represented by a single value of the integral. If this would be the cause of slow convergence, it might be advantageous to define 10 independent Max2 constraints, each defined only on one finite time element.

Unfortunately, neither of the approaches has performed significantly better than the original.

To conclude, if it is possible to use the slack method with a sufficiently high degree of the approximation, the results will be obtained with a high accuracy, without constraint violation and in a relative low number of iterations. However, for the reasons given above, it is not a method for a general use.

Chapter 4

Optimisation of a Waste-Water Treatment Plant

4.1 Problem Definition

We consider a problem of energy resources minimisation for a small-size waste-water treatment plant (WTP). A fairly detailed description of the plant model can be found in Chachuat et al. (2001). The model can be characterised as a hybrid continuous dynamic system described by two sets of 11 differential equations for aerobic/anaerobic conditions of the form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}^{(1)}(\boldsymbol{x}) \qquad \text{(aeration periods)} \tag{4.1}$$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}^{(2)}(\boldsymbol{x}) \qquad \text{(non-aeration periods)} \tag{4.2}$$

where $\boldsymbol{f}^{(1)}$ and $\boldsymbol{f}^{(2)}$ are the corresponding right hand sides of the differential equations. They differ in the equation describing dissolved oxygen saturation concentration x_O where the equation for the aeration periods contains one additional term e

$$\frac{\mathrm{d}x_O}{\mathrm{d}t} = f_O^{(1)}(\boldsymbol{x}) + e(\boldsymbol{x}) \qquad \text{(aeration periods)} \tag{4.3}$$

$$\frac{\mathrm{d}x_O}{\mathrm{d}t} = f_O^{(1)}(\boldsymbol{x}) \qquad \text{(non-aeration periods)} \tag{4.4}$$

The primary source of disturbantions are daily variations of influent flowrate and organic load modelled by a typical diurnal pattern.

In order to simplify the model, it is transformed into a non-hybrid model described by 11 ordinary differential equations of the form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}^{(1)}(\boldsymbol{x}) + u_1 e(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}, u_1) \tag{4.5}$$

where the input u_1 is a sequence switching between 1 and 0 and represents the state of turbines (on/off) that aerate the plant. Without loss of generality, it is assumed that at time $t = 0$ the turbines are on and that the initial state at $t = 0$ is known and given by $\boldsymbol{x}_0$.

The real manipulated variable that influences the operation of WTP is the sequence of switching times, i.e. times when the turbines switch on/off. It is possible to normalise the model with respect to time to obtain its alternative description where the true manipulated variable occurs in the system equations. Let us assume that there are N_c aeration/non-aeration cycles within one day and let us denote the lengths of the finite time elements by $\Delta t_1, \ldots, \Delta t_{2N_c}$. The final time of the simulation/optimisation (one day) is given as

$$T = \Delta t_1 + \Delta t_2 + \cdots + \Delta t_{2N_c} \tag{4.6}$$

The aim of the normalisation is to change the time intervals Δt_1, $\Delta t_1 + \Delta t_2$, ..., T into evenly spaced fixed time intervals $1/n, 2/n, \ldots, 1$. This results in the modified system equations

$$\frac{\mathrm{d}\dot{\boldsymbol{x}}}{\mathrm{d}\tau} = u_2 \boldsymbol{f}\left(\boldsymbol{x}, u_1\right), \quad \tau \in [0, 1] \tag{4.7}$$

where u_2 is a piece-wise constant sequence of the length $2N_c$ containing the switching times $\Delta t_1, \ldots, \Delta t_{2N_c}$.

The goal of the optimisation is to minimise the energy consumption of aeration. This is equivalent to minimisation of the total aeration time during one day. The cost function can be expressed as

$$\min_{u_2} J_0 = \frac{\sum_{j=1}^{2N_c} u_2^{\text{odd}}}{\sum_{j=1}^{2N_c} u_2^j} \tag{4.8}$$

In addition, several constraints are specified that have to be satisfied during the operation of the plant. These include maximal allowed residual concentrations and the bounds on the optimised time lengths to ensure the feasibility of the computed aeration profiles.

The constraints that are usually active during the operation are as follows:

$$\mathrm{TN}_{\max} \leq 10\,\mathrm{mg/L}, \quad \text{(total nitrogen)} \tag{4.9}$$

$$u_2^{\text{odd}} \in [15, 120]\,\mathrm{min}, \quad \text{(aeration periods)} \tag{4.10}$$

$$u_2^{\text{even}} \in [30, 120]\,\mathrm{min}, \quad \text{(non-aeration periods)} \tag{4.11}$$

The time limits are defined to avoid too frequent cycling of the turbines (minimum), to prevent floc sedimentation and modification of the degradation performances (maximum).

Finally, the constraint of one day operation can be expressed as

$$u_2^1 + u_2^2 + \cdots + u_2^{2N_c} = 68400\,\mathrm{s} \tag{4.12}$$

4.2 Problem Solution

Among the techniques described in the previous chapter that solve inequality state path constraints, the Max2 approach has been chosen. Unfortunately, it is not possible to use the slack variable approach as the manipulated variable is strictly piece-wise constant and thus cannot be transformed into a new state variable.

The number of cycles has been fixed to $N_c = 29$ and was not a subject of further optimisation as this would lead to a mixed integer dynamic optimisation. Again, the dynamic optimisation package DYNO was employed (Fikar, 2001).

The solution statistics shown in the Table 4.1. Several tolerance levels are given and the initial guess at the tighter tolerances was the final solution in the previous line. This is called the *sequenced initial guess method*. The rationale behind is that at the looser tolerances, the integration of the system and adjoint equations is much faster as at the tighter tolerances. The start of the optimisation is far away from the optimum and thus the precision of gradients is not so crucial. The required time for one iteration varies between 3 seconds with the loosest tolerances to several minutes with the tightest tolerances.

We can see from Table 4.1 that an approximate minimum can be found very quickly with loose tolerances. However, as the cost function minimised appears to be quite flat near the optimum (which is bounded from below by 30% due to the operational constraints) different aeration strategies can be found that yield similar values of the cost function. For example, Figures 4.1, 4.2 show the nitrogen concentration during the day as well as the corresponding aeration rates for the loosest and the tightest tolerances used. The aeration rate shown in the

Table 4.1: Summary of the results for 29 cycles per day

$J[\%]$	Precision (IVP, NLP)	Iterations (IVP, NLP)
32.34	$10^{-7}/10^{-5}$	208/157
31.68	$10^{-9}/10^{-7}$	429/290
31.48	$10^{-11}/10^{-9}$	251/182
31.26	$10^{-13}/10^{-11}$	338/278

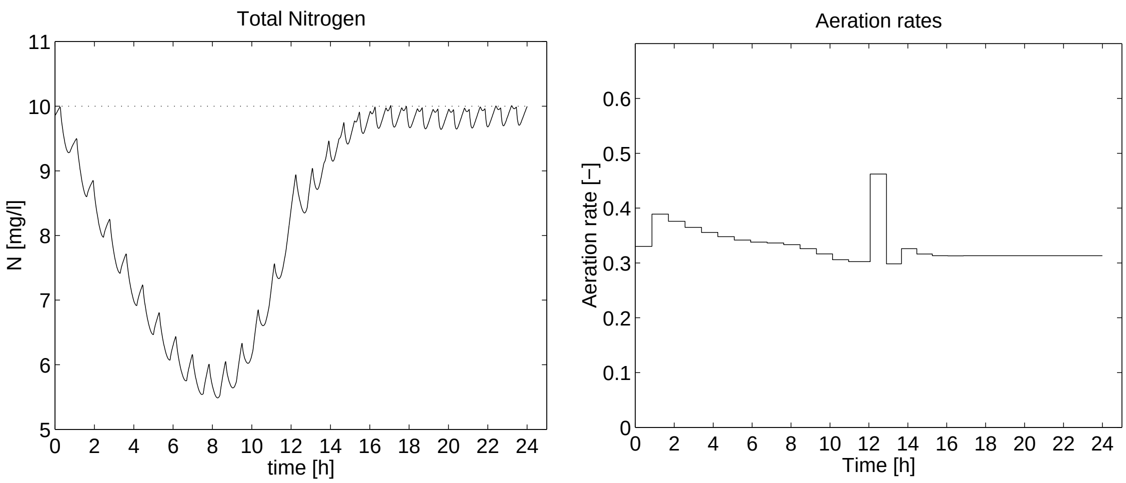

Figure 4.1: Optimal trajectories for $J = 32.34\%$ (the loosest precisions). Left: Nitrogen constraint, right: aeration policy

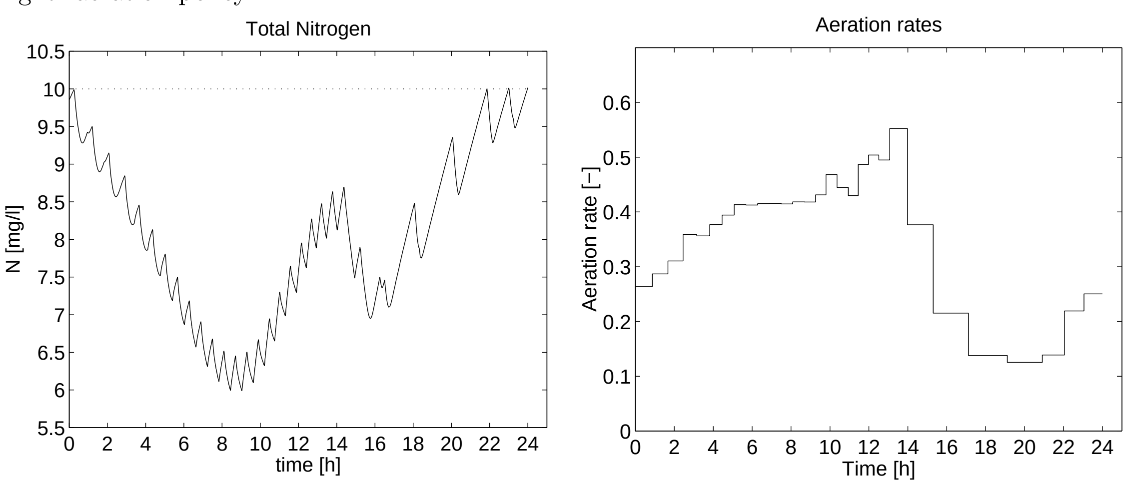

Figure 4.2: Optimal trajectories for $J = 31.26\%$ (the tightest precisions). Left: Nitrogen constraint, right: aeration policy

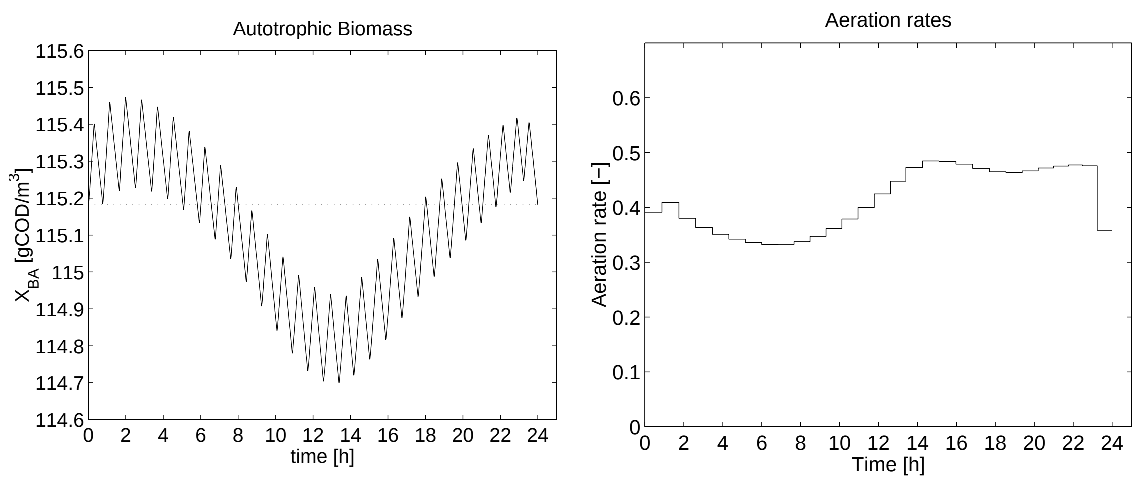

Figure 4.3: Optimal aeration policy respecting the biomass constraints. Left: Autotrophic biomass concentration, right: aeration policy

figures is defined as the percentage of aeration time during one aeration/non-aeration cycle. It can be seen that only small differences in the cost function value can produce the trajectories entirely different.

Table 4.1 shows also, that the sequenced initial guess approach yielded the number of iterations approximately the same for different precisions.

The obtained value of the cost function is very close its lower bound and much smaller than the values reported at the real plant. This leads to suspect, that the regime obtained may violate the permanent regime of the plant. Indeed, simulating this profile for several weeks indicates that the concentration of the biomass gradually decreases and causes violation of the maximum nitrogen concentration.

This problem is caused by the fact, that the notion of the permanent regime has not been correctly formulated in the definition of the optimisation problem. One possible improvement is to add final time constraints that the final state should be equal to the initial state:

$$\boldsymbol{x}(1) = \boldsymbol{x}_0 \tag{4.13}$$

However, this problem converges very slowly, even if the constraints have been normalised and converted into squared inequalities of the form

$$\varepsilon - \left(\frac{x_i(1) - x_{i,0}}{x_{i,0}} \right)^2 \geq 0 \tag{4.14}$$

Actually, not all states have to be under constraints at the final time. In fact, final nitrogen concentration has already been fixed with the Max2 constraint. Therefore, only the autotrophic and heterotrophic biomass concentrations have been constrained yielding the optimum aeration fraction of 41.3%. The optimal aeration policy is shown in Fig. 4.3 with the autotrophic biomass concentration on the left side. The heterotrophic biomass shows the same type of the trajectory.

The obtained results give the minimum possible aeration rates for the given plant and provide several outcomes. The aeration rate can be compared with the one actually used at the real plant and helps to determine whether there is a room for an improvement over the existing aeration strategies and whether it is reasonable to make investments. Moreover, the aeration during the day can serve as setpoint trajectory at the real plant.

Chapter 5

Conclusions

This report investigated several approaches to state path constrained dynamic optimisation problems. In its first part, an overview of existing methods has been given.

The second part has been devoted to actual comparison of the methods as well as to propositions for their improvement. One can conclude, that the slack variable approach is very powerful and converges quickly to optimum. However, it is not very suitable for a general type of the problem as in principle it is not able to handle control variable constraints in the presence of state inequality constraints.

In the category of the general all-purpose methods, the best approach was to convert the original inequality constraint into equality on additional final state using the squared max operator.

In the last part, optimisation of waste-water treatment plant has been considered. It has been shown, how to rewrite and normalise the original hybrid model to the standard form that can be used in general dynamic optimisation packages.

The goal of the optimisation was to minimise energy consumed by aeration turbines subject to new effluent constraints specified by the EU. Optimisation over one day has been considered. The results obtained indicate that there is a large variation of the aeration strategies that yield approximately the same value of the cost function. However, one has to take care in defining the optimisation problem as the dynamics of biomass is very slow. If that is not taken into account, aeration policies can be obtained that will in the long rung destabilise the usual mode of operation. Therefore, a modified optimisation problem has been defined and it has been shown that the biomass concentration does not decrease.

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