POLYNOMIAL APPROACH TO CONSTRAINED RECEDING HORIZON PREDICTIVE CONTROL

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Abstract: This contribution addresses an alternative stability proof of the Constrained Receding Horizon Predictive Control (CRHPC) problem based on the closed-loop design.
The second aim of this paper is to show that the actual horizon lengths are too conservative. A modified method that zeroes only the tracking error while relaxing the assumption about future control increments is derived. Again, the stability proof is performed directly in closed-loop and not as usually, in open-loop.
Finally, it is shown how to rewrite the proposed controller within the predictive control framework.

Keywords: predictive control, stability, algebraic theory

1. INTRODUCTION

Generalised Predictive Control (GPC) proposed by Clarke *et al.* (1987) has been accepted in academia and also applied widely in industry. For safety reasons, the stability of the closed-loop has to be assured.

As GPC is based on receding finite horizon minimisation, it inherently suffers from stability problems. The original proofs deal mainly with limiting cases. Stability results for reasonable horizon lengths have been reported that use terminal constraints by Clarke and Scattolini (1991)(CRHPC), Mosca and Zhang (1992)(SIORHC), Rossiter and Kouvaritakis (1993)(SGPC) where it is required that both the control and tracking error are finite-time responses. Actually, in the last reference, the authors show the equivalence of all these methods. Another approach follow Fikar and Engell (1997)(YKPC) and choose the predictive controller from the set of controllers given by Youla-Kučera (YK) parametrisation. It has been shown that terminal constraints are automatically satisfied and need not explicitly be used.

The purpose of this contribution is to tackle the stability problem and the choice of horizon lengths directly from the closed-loop point of view. At first, an alternative proof of stability of CRHPC will be given. This is based on algebraic theory and dead-beat closed-loop systems. The main idea is to find such a controller that equals to predictive controller with no degrees of freedom while retaining stability.

The second aim of this paper is to propose a modified method that reduces minimum horizon lengths. Again, a closed-loop solution is searched for that is stable. Some assumptions of the original method are shown to be superfluous.

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This paper is organised as follows. The original CHRPC method is given in the Section 2. Section 3 re-derives the method using an algebraic approach. In the Section 4 the main results are presented and a modified method is derived. The Section 5 discusses properties of the proposed method and shows some simulation results. Finally, the Section 6 provides the conclusions.

1.1 *Notation*

All systems in this contribution are assumed to be single input single output, linear, time-invariant, and discrete-time. The systems are described by means of fractions of polynomials in an indeterminate z^{-1}, used in the z-transform and normally interpreted as delay operator. The reader is referred to Kučera (1979) whose notations are adopted hereafter as much as possible.

For simplicity, the arguments of polynomials are omitted whenever possible - a polynomial $X(z^{-1})$ is denoted by X. We define the adjoint of a polynomial X as $X^*(z^{-1}) = X(z)$. Further, for any polynomial X, we define $\langle X \rangle$ as the coefficient of z^0, i.e. the constant term. Any polynomial X can be factored as X^+X^- where X^+ denotes its stable and X^- its totally unstable (anti-stable) part. The greatest common divisor of two polynomials X, Y is denoted by (X, Y).

2. CRHPC

Let us consider a discrete-time plant with input-output representation of the form

$$Ay = Bu, \tag{1}$$

where y, u are the process output and manipulated input sequences, respectively. A and B are polynomials in z^{-1} that describe the input-output properties of the plant and $(A, B) = 1$. It is assumed that $A(0) \neq 0$ and $B(0) = 0$ (all delays are included in B).

The optimal control problem considered is the receding horizon control of (1) stated as follows.

Problem 1. Find such a sequence of control increments $\tilde{u}(t), \tilde{u}(t+1), \ldots, \tilde{u}(t+N_u)$ that minimises the cost function

$$J(\tilde{u}) = \psi \sum_{i=1}^{N} e(t+i)^2 + \phi \sum_{i=0}^{N_u} \tilde{u}(t+i)^2 \tag{2}$$

subject to constraints

$$\begin{aligned} & y(t+N+j) = w(t+N) \\ & \quad j = 1, \ldots, m \end{aligned} \tag{3}$$

$$\begin{aligned} & \tilde{u}(t+N_u+j) = 0 \\ & \quad j = 1, \ldots, N - N_u + m, \end{aligned} \tag{4}$$

where $w(t)$ is the reference sequence, ψ and ϕ are weighting factors, and $e(t+j) = w(t+j) - y(t+j)$ represents the sequence of tracking errors. N, N_u are output and control horizon, respectively. Only the first control increment calculated $\tilde{u}(t)$ is to be acting at the plant input.

The purpose of the constraints is to assure that the plant output will be at a desired set-point not only within the interval $[t+N+1, t+N+m]$ but also afterwards. Therefore, the sequence of control increments is also constrained in such a way that also after $t+N+m$ the increments will be zero.

The solution of this problem given by Clarke and Scattolini (1991), Mosca and Zhang (1992) states in principle the following:

Theorem 1. The closed-loop is asymptotically stable for $\psi \geq 0, \phi > 0$ and if

$$N = \deg(B) - 1 + n \tag{5}$$

$$N_u = \deg(A) + n, \; n = 0, 1, 2, \ldots \tag{6}$$

$$m = \max(\deg(A) + 1, \deg(B)) \tag{7}$$

(1) If $n = 0$ then the method yields a stable state dead-beat closed-loop system,
(2) The control law is of the form

$$P\tilde{u}(t) = Rw(t+N) - Qy(t) \tag{8}$$

where $\deg(P) = \deg(B) - 1$, $\deg(Q) = \deg(A)$, $\deg(R) = N - 1$.
(3) If $n = 0$ then $\deg(R) = 0$.

Note: The last item in the Theorem is not so obvious. When the CRHPC yields dead-beat response, it behaves like GPC with $N_1 = N$ so that the first N_1 control error steps are not penalised. Therefore, the reference sequence $w(t+1), \ldots, w(t+N)$ does not occur in the control-law (8). In the remaining steps from $t+N+1$ to $t+N+m$ it is required that the reference is constant and equal to $w(t+N)$. Hence, the control law (8) contains only the term $w(t+N)$ and R is a constant.

3. ALTERNATIVE STABILITY PROOF OF CRHPC

Let us now show the same but from another direction. The idea is to show that a dead-beat controller will yield the CRHPC control law. As assumptions, we will take the conclusions from Theorem 1.

3.1 *Closed-loop Configuration*

We assume that the reference w is generated via

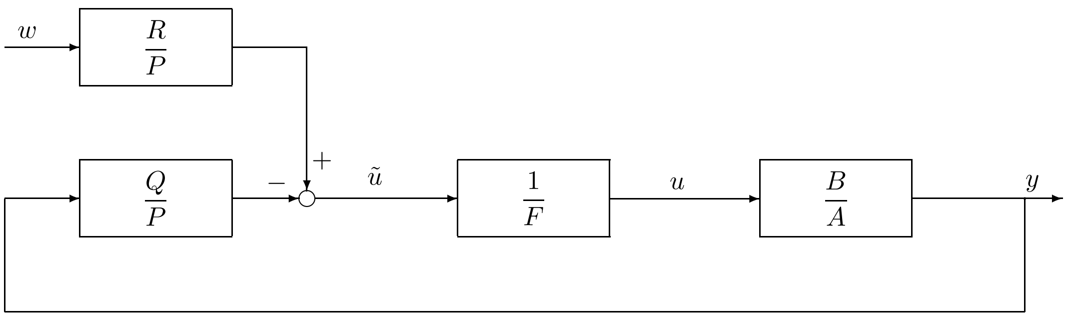

Fig. 1. 2DoF control configuration with explicit integral action

$$Fw = G, \tag{9}$$

where $(F, G) = 1$. A further assumption is that (A, F) are coprime, thus $(A, F) = 1$.

The 2DoF controller is another dynamical system described by the equation

$$P\tilde{u} = Rw - Qy, \tag{10}$$

where P, Q, R are controller polynomials that are coprime and $P(0)$ is nonzero. In addition, an integrator that forms a part of the controller is used in the form

$$\tilde{u} = Fu \tag{11}$$

to track the class of references given above. When assuming the usual class of references, namely step changes, then $F = 1 - z^{-1}$, $G = 1$ and the signal $\tilde{u} = \Delta u$ is a sequence of control increments.

This description of the closed-loop configuration is shown in Fig. 1.

3.2 *General 2DoF Controller*

Given a stable closed-loop polynomial M, the minimum degree controller that internally stabilises the closed-loop system is defined uniquely and is given as follows:

Theorem 2. The minimum degree controller P, Q, R is given as a solution of two pairs of Diophantine equations that minimise the degrees Q, R

$$\begin{aligned} AFP + BQ &= M, \\ FS + BR &= M. \end{aligned} \tag{12}$$

Proof. Kučera (1991). □

Any linear, time-invariant controller corresponding to Fig. 1 is among the controllers given by Theorem 2. The CRHPC controller must generate a closed-loop control law as given by (8). Therefore, the assumptions leading to the CRHPC are as follows:

(1) Finite length sequences $\tilde{u}, e \Rightarrow M = 1$,
(2) Minimum degree solution of the Diophantine equations (12).

These assumptions equate the general structure of the 2DoF controller to the CRHPC controller with one exception. The CRHPC controller operates with future reference signal $w(t + N)$ and not with the past one as here. This can be obtained either by adding the term z^N to the forward path as it was done by Grimble (1997), or simply by pretending that the signal $w(t)$ is actually $\bar{w}(t + N)$. In any case, this can be done without loss of generality as tracking and regulation objectives are independent and the important is regulation only.

The sequences $\tilde{u}, e$ are obtained from the closed-loop equations as

$$\tilde{u} = ARG, \quad \deg \tilde{u} = \deg(A), \tag{13}$$

$$e = SG, \quad \deg e = \deg(B) - 1. \tag{14}$$

Therefore, the minimum horizons that yield stable closed-loop system must at least contain all non-zero coefficients of $\tilde{u}, e$ and are given as

$$N = \deg(B) - 1, \tag{15}$$

$$N_u = \deg(A). \tag{16}$$

The horizons are the same as in Theorem 1 and the controller is unique. Therefore, the controllers are identical and have the same stability properties.

4. NEW METHOD

To force the tracking error identically to zero after some horizon, one does not have to employ the state dead-beat strategy. The constraint on the future control increments can be relaxed which may give a shorter output horizon as before. The corresponding closed-loop problem is as follows:

Problem 2. Find such a controller P, Q, R that the closed-loop system is BIBO stable and the control error $e = w - y$ is a polynomial of minimum degree.

The solution of the problem is given below.

Theorem 3. The 2DoF closed-loop finite sequence control error problem has a solution $e = E$. The unique solution is given by

$$E = S_1 G^-, \ \deg(E) \leq \deg(B^- G^-) - 1 \quad (17)$$

The feedback controller Q/P is unique and is given as the solution of the Diophantine equation that minimises the degree of Q

$$AFP + BQ = B^+ G^+. \quad (18)$$

The feedforward part of the controller is calculated from the second Diophantine equation

$$FS_1 + B^- R = G^+ \quad (19)$$

as the solution that minimises the degree of S_1.

Proof. Inspecting Fig. 1, the sequence $\tilde{u}$ can be written as

$$\tilde{u} = \frac{ARG}{AFP + BQ} = \frac{ARG}{M} \quad (20)$$

and the control error sequence as

$$e = \left(1 - \frac{BR}{M}\right) \frac{G}{F} \quad (21)$$

In order to cancel unstable modes of F from the denominator, the following Diophantine equation must hold

$$F^- S + BR = M. \quad (22)$$

This yields for e

$$e = \frac{SG}{MF^+} = \frac{SG}{(F^- S + BR)F^+} \quad (23)$$

and e should be a polynomial. In order to cancel F^+ from the denominator, F^+ must be a factor of S, thus $S = F^+ \tilde{S}$. This gives

$$e = \frac{\tilde{S}G}{F\tilde{S} + BR}. \quad (24)$$

Further reduction is possible if

$$\tilde{S} = S_1 B^+. \quad (25)$$

The control error signal thus becomes

$$e = \frac{S_1 G}{FS_1 + B^- R}. \quad (26)$$

Now, the only possibility to reduce e to a polynomial of the smallest degree and still leave the closed-loop polynomial stable is to cancel the stable zeros of G in the numerator by the denominator

$$FS_1 + B^- R = G^+ \quad (27)$$

leading to (17). This Diophantine equation has a solution iff $(B^-, F) = 1$ which is assured by assumptions. The solution that minimises the degree of S_1 is searched.

The closed-loop polynomial M is given as

$$M = F^- S + BR = B^+ G^+ \quad (28)$$

and is stable.

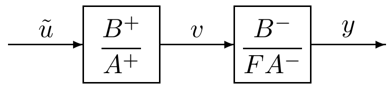

Fig. 2. The role of the intermediate signal v

The feedback part of the controller is given by

$$AFP + BQ = M \quad (29)$$

The minimum degree solution for Q is searched so that the controller is in a a minimum realisation.

The proof is completed by checking stability of the signal $\tilde{u}$ for which holds

$$\tilde{u} = \frac{ARG}{M} = \frac{ARG^-}{B^+} \quad (30)$$

and we see again that its stability is assured. □

What follows from this theorem, are the minimum horizon lengths. Assuming step references ($G = 1, F = 1 - z^{-1}$), the output horizon has been shortened and is equal to the number of unstable zeros of the plant:

$$N = \deg(B^-) - 1 \quad (31)$$

The situation is more complicated with the control increments. The predictive method must provide a stable sequence $\tilde{u}$ given by (30). Therefore, the optimised variable will be only a polynomial part of $\tilde{u}$ given by (30). The natural choice is the whole numerator $v_1 = B^+ \tilde{u}$ with degree $\deg(A)$. However, as it was shown in Rawlings and Muske (1993), Fikar and Kučera (2000), the minimum number of optimised variables that leads to a stable closed-loop system is equal to the number of unstable plant poles. For this purpose, let us decompose $\tilde{u}$ into stable and anti-stable parts

$$\tilde{u} = \frac{A^+}{B^+} A^- RG^- = \frac{A^+}{B^+} v. \quad (32)$$

Here, v is a polynomial with degree $\deg(A^-)$. Its physical interpretation is shown in Fig. 2 and it is clear that it is an intermediate control signal that acts only on the unstable part of the controlled system. As in CRHPC, v has to satisfy the equality constraint

$$\begin{aligned} v(t + N_u + j) &= 0 \\ j &= 1 \ldots N - N_u + m, \end{aligned} \quad (33)$$

where

$$N_u = \deg(A^-). \quad (34)$$

The stabilising predictive control strategy may now be defined as follows

Theorem 4. The predictive control strategy defined in Problem 1 is asymptotically stabilising if the horizons are set as

$$N = \deg(B^-) - 1 + n, \tag{35}$$

$$N_u = \deg(A^-) + n, \; n = 0, 1, 2, \ldots \tag{36}$$

$$m = \max(\deg(A^-) + 1, \deg(B^-)). \tag{37}$$

The sequence of control increments is generated from

$$\tilde{u} = \frac{A^+}{B^+} v \tag{38}$$

and v instead of $\tilde{u}$ is the optimised variable. The loop is closed by applying the first element $v(t) = \tilde{u}(t)$ to the system.

Note: In the original CRHPC proof it is required that the control weights ϕ must be greater than zero. However, the closed-loop 2DoF controller (either state or tracking error dead-beat) is for minimum horizons unique and does not depend on the weights. Therefore, it can be concluded, that the condition $\phi > 0$ can be omitted. Of course, if the horizons are larger than minimal, a well posed optimisation problem requires either ϕ or ψ be greater than zero. Actually, the condition $\phi > 0$ comes from continuous-time LQ formulation where it is necessary. It can be omitted in the discrete-time formulation.

4.1 *Implementation*

The actual implementation of the proposed predictive method must take into account that the optimised variable is no longer $\tilde{u}$ but v. The relation between them is given by (32) and can be rewritten into matrix notation as

$$\boldsymbol{T}_b \begin{pmatrix} \tilde{u}_t \\ \vdots \\ \tilde{u}_{t+N+m} \end{pmatrix} = \boldsymbol{T}_a \begin{pmatrix} v_t \\ \vdots \\ v_{t+N_u} \end{pmatrix}, \tag{39}$$

where $\boldsymbol{T}_a, \boldsymbol{T}_b$ are Toeplitz matrices with dimensions $[N+m, N_u+1]$ and $[N+m, N+m]$, respectively. These matrices contain coefficients of the polynomials A^+, B^+ columnwise. For example $\boldsymbol{T}_b$ is given as

$$\boldsymbol{T}_b = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ b_1^+ & 1 & 0 & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \ldots & 1 \end{pmatrix}. \tag{40}$$

As usual in predictive control, the output predictions can now be expressed as

$$\hat{\boldsymbol{y}} = \boldsymbol{G}\tilde{\boldsymbol{u}} + \boldsymbol{f} \tag{41}$$

$$= \boldsymbol{G}\boldsymbol{T}_b^{-1}\boldsymbol{T}_a \boldsymbol{v} + \boldsymbol{f} \tag{42}$$

$$= \boldsymbol{G}_1 \boldsymbol{v} + \boldsymbol{f} \tag{43}$$

Therefore, the derivation of the method remains the same as in CRHPC with the only change that the matrix $\boldsymbol{G}$ is changed for $\boldsymbol{G}_1$ in all computations.

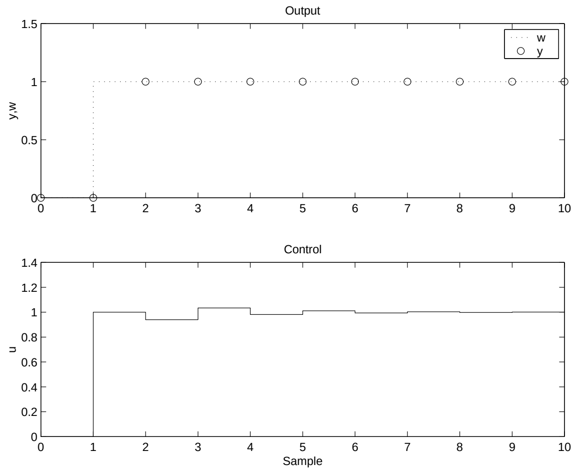

Fig. 3. Control of the stable 2nd order system

It can be shown that the matrix $\boldsymbol{T}_b$ is invertible and reduces to the identity matrix if the system numerator is strictly unstable.

5. DISCUSSION

The proposed method reduces significantly the minimal horizons when the predictive controller leads to a stable closed-loop behaviour. Of course, the limiting case is not very suitable from the point of view of performance. As only the tracking error is constrained to be a polynomial of the smallest degree and the control increments are a stable sequence, intersampling oscillations are likely to occur when controlling continuous-time systems.

To show the behaviour of the method, the following stable system is considered

$$B = (1 + 0.56z^{-1})z^{-1}, \tag{44}$$

$$A = (1 + 0.3z^{-1})(1 + 0.2z^{-1}). \tag{45}$$

Hence, $B^- = z^{-1}$, $A^- = 1$. The horizons according to Theorem 4 are given as $N = 0, N_u = 0, m = 1$. The results are shown in Fig. 3 and confirm the theoretical expectations.

6. CONCLUSIONS

This contribution has investigated a problem of alternative ways to prove stability of receding horizon schemes. The idea behind the proofs is to derive the method when the loop is already closed and to find the minimum degree controller that leads to stability. The method CRHPC has been re-derived in this sense. By carefully investigating the assumptions, it was found that dead-beat formulation of the method is too restrictive.

Therefore, the assumption about finite length control was relaxed yielding horizon lengths that are smaller than those of the original method. Also, as the control increments are no longer finite length sequence, the feasibility properties are improved.

The second aim of was to rewrite the proposed 2DoF controller into the equivalent form of a predictive controller. A new optimised signal was found that has only a finite number of terms in spite of the corresponding stable sequence of the control increments.

The closed-loop stability proof shows also, that the usual conditions about positive weightings in the cost function may be relaxed.

Acknowledgments

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