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RELATIONS BETWEEN STABLE PREDICTIVE CONTROL AND POLE PLACEMENT

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Abstract: In this paper, several relations between pole placement (PP) controller and unconstrained predictive controller are investigated. The placement of poles at certain positions leads to various types of time optimal control and has a direct relation to the choice of minimum control and output horizons for stabilising MPC. On the other hand, a predictive control method for linear systems is derived that can be turned into some well known stabilising MPC methods. This MPC method is closely related to the PP design.

Keywords: Predictive control, stability, algebraic systems theory.

1. INTRODUCTION

Generalised Predictive Control (GPC) proposed by (Clarke *et al.*, 1987) is nowadays an accepted control method capable to deal with constraints and difficult processes. To assure stability, a constraint on terminal states is to be imposed (Clarke and Scattolini, 1991; Mosca and Zhang, 1992; Rossiter and Kouvaritakis, 1993). It is well known that in the absence of inequality constraints the method generates a linear time-invariant controller.

The main idea of this paper is to show some relations between the stable predictive control design and pole placement.

Two approaches will be investigated. In the first one, closed-loop expressions for a stable two-degree-of-freedom controller are derived. Then, locations of closed-loop are examined that lead to time optimal control and the relations to predictive control.

In the second approach, no degrees of freedom for the predictive controller will be assumed. The further assumption is to use an unconstrained controller. These assumptions will lead to a unique predictive control law. In the next step, a pole placement design will be invoked and it will be shown, that many well known predictive control schemes can be obtained for a particular closed-loop pole locations.

Finally, a cost function will be specified, that does not change the properties of the predictive controller when the degrees of freedom for optimisation change. This cost can be used if the performance is to be specified via the closed-loop poles.

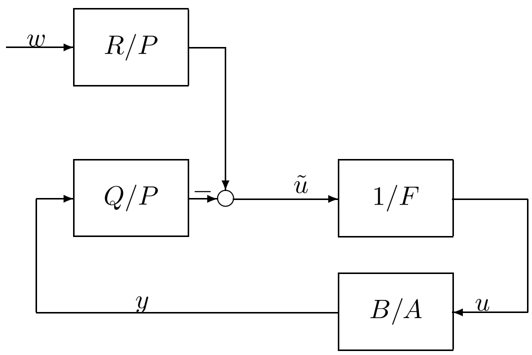

Fig. 1. 2DoF control configuration with explicit integral action

2. CLOSED-LOOP SYSTEM

Let us consider a discrete-time plant with input-output representation of the form

$$Ay = Bu, \tag{1}$$

where y, u are the process output and manipulated input sequences, respectively. A and B are polynomials in z^{-1} that describe the input-output properties of the plant and (A, B) are coprime. It is assumed that $A(0) \neq 0$ and $B(0) = 0$ (all delays are included in B).

We assume that the reference w is generated via

$$Fw = G, \tag{2}$$

where (F, G) are coprime.

The 2DoF controller is another dynamical system described by the equations

$$P\tilde{u} = Rw - Qy, \;\; \tilde{u} = Fu, \tag{3}$$

where P, Q, R are controller polynomials that are coprime and $P(0)$ is nonzero. In addition, an integrator forms a part of the controller to track the class of references given above. For the purpose of predictive control, the usual class of references – step changes will be assumed. Then $F = 1 - z^{-1}$, $G = 1$ and the signal $\tilde{u} = \Delta u$ is a sequence of control increments.

This description of the closed-loop configuration is shown in Fig. 1.

Given a stable closed-loop polynomial M, the minimum degree controller that internally stabilises the closed-loop system is defined uniquely and is given as follows:

Theorem 1. The minimum degree controller P, Q, R is given as a solution of two Diophantine equations that minimise the degrees Q, R

$$\begin{aligned} AFP + BQ &= M, \\ FS + BR &= M. \end{aligned} \tag{4}$$

PROOF. Kučera (1979). □

The general controller is uniquely characterised by the choice of the closed-loop polynomial M. Some choices of M that are important in time optimal control are given below:

Theorem 2. Consider step changes in references and the following closed-loop time optimal control problems for which BIBO stability is to be guaranteed:

(1) State dead-beat: Minimum number of nonzero steps of control error, finite number of nonzero steps of control increments,
(2) Control dead-beat: Minimum number of nonzero steps of control increments and stable control error,
(3) Control error dead-beat: Minimum number of nonzero steps of control error and stable control increments.

The corresponding closed-loop polynomials and minimum degrees of relevant signals with 2DoF controller are

$$\begin{aligned} M_1 = 1,\;& \deg(e) = \deg(B) - 1, \\ & \deg(\tilde{u}) = \deg(A), \end{aligned} \tag{5}$$

$$M_2 = A^+,\; \deg(\tilde{u}) = \deg(A^-), \tag{6}$$

$$M_3 = B^+,\; \deg(e) = \deg(B^-) - 1. \tag{7}$$

where A^+, B^+ denote stable and A^-, B^- strictly unstable parts of the respective polynomials. The minimum number of nonzero steps of a signal is then given as one plus the respective polynomial degree.

PROOF.

(1) (Kučera, 1979),
(2) (Fikar and Kučera, 2000),
(3) (Fikar and Unbehauen, 1999). □

The optimal pole locations are important for predictive control and have close relation to minimum possible horizons that produce stable closed-loop. It is well known that the predictive controllers have the structure of a 2DoF controller. If $M = A^+$, the minimum control horizon is equal to the number of unstable system poles $\deg(A^-) + 1$ where the additional unstable pole comes from the integrator F. Correspondingly, the minimum output horizon cannot be smaller than the number of unstable system zeros which corresponds to the case of $M = B^+$.

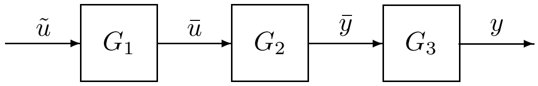

Fig. 2. System decomposition

3. PREDICTIVE CONTROLLER

Predictive controllers usually operate on the signals $\tilde{u}, e = w - y$. These are given from (1)–(4) as

$$\tilde{u} = \frac{ARG}{M},\ e = \frac{SG}{M}. \tag{8}$$

If the predictive controller is to be equivalent to the nominal controller, it has to generate the signals $\tilde{u}, e$ from its internal signals $\bar{u}, \bar{e}$ by filtering through the term $1/M$. Moreover, the requirement of closed-loop stability invoked with the concept of state terminal constraints requires that the signals $\bar{u}, \bar{e}$ are polynomials of finite length. This suggests the relations

$$\tilde{u} = \frac{\bar{u}}{M}, \quad \deg(\bar{u}) = \deg(A), \tag{9}$$

$$e = \frac{\bar{e}}{M}, \quad \deg(\bar{e}) = \deg(B) - 1. \tag{10}$$

Clearly, the degrees of $\bar{u}, \bar{e}$ are related here to the state dead-beat controller. However, further reduction of $\bar{u}, \bar{e}$ is possible. This can be achieved by decomposing the controlled system to stable and anti-stable parts. Next, state dead-beat is applied to the unstable part. The result is given in Fig. 2 with

$$G_1 = \frac{M}{A^+},\ G_2 = \frac{B^-}{FA^-},\ G_3 = \frac{B^+}{M}, \tag{11}$$

hence we have the following result:

Theorem 3. Let us define the following signals

$$\tilde{u} = \frac{A^+}{M}\bar{u}, \quad \deg(\bar{u}) = N_u, \tag{12}$$

$$y = \frac{B^+}{M}\bar{y}, \quad \bar{e} = \bar{w} - \bar{y}, \quad \deg(\bar{e}) = N. \tag{13}$$

and horizons

$$N_u = \deg(A^-), \tag{14}$$

$$N = \deg(B^-) - 1, \tag{15}$$

$$m = \max(\deg(A^- F), \deg(B^-)). \tag{16}$$

The nominal controller given by the closed-loop polynomial M is equivalent to the predictive controller with no degrees of freedom given by the set of equality constraints

$$\bar{y}_{t+N+j} = \bar{w}_{t+N},\ j = 1, \ldots, m \tag{17}$$

$$\bar{u}_{t+N_u+j} = 0,\ j = 1, \ldots, N - N_u + m, \tag{18}$$

The internal sequence of control increments $\bar{u}$ is calculated from

$$\boldsymbol{G}_1 \bar{\boldsymbol{u}} = \bar{\boldsymbol{w}}_1 - \bar{\boldsymbol{f}}_1, \tag{19}$$

where

$$\bar{\boldsymbol{y}}_1 = \boldsymbol{G}_1 \bar{\boldsymbol{u}} + \bar{\boldsymbol{f}}_1, \tag{20}$$

$$\bar{\boldsymbol{u}}^T = (\bar{u}_t\ \bar{u}_{t+1}\ \ldots\ \bar{u}_{t+N_u-1}), \tag{21}$$

$$\bar{\boldsymbol{y}}_1^T = (\bar{y}_{t+N}\ \bar{y}_{t+N+1}\ \ldots\ \bar{y}_{t+N+m-1}), \tag{22}$$

$$\bar{\boldsymbol{f}}_1^T = (\bar{f}_{t+N}\ \bar{f}_{t+N+1}\ \ldots\ \bar{f}_{t+N+m-1}), \tag{23}$$

$$\bar{\boldsymbol{w}}_1^T = (1\ \ldots\ 1) M(1)/B^+(1) \tag{24}$$

$$\boldsymbol{G}_1 = \begin{pmatrix} g_N & \cdots & g_{N-m+1} \\ \vdots & \ddots & \vdots \\ g_{N+m-1} & \cdots & g_N \end{pmatrix}. \tag{25}$$

The actual control increment $\tilde{u}(t)$ is calculated from (12) and applied to the controlled system in the receding horizon manner.

PROOF. For the minimum degrees of $\bar{u}, \bar{e}$ consider state dead-beat control of the unstable system B^-/FA^-. From Theorem 2 follow the relations (14), (15). To force this dead-beat strategy in the predictive controller, the sequences $\bar{u}, \bar{e}$ are required to be zero in (N, ∞), (N_u, ∞), respectively. To assure this, define the state dimension m by (16). Then, the set of constraints (17), (18) guarantees zero sequences in the whole desired interval.

The unstable part B^-/FA^- of the system is linear and has the input $\bar{u}$ and output $\bar{y}$. Hence, the equation (20) holds. The corresponding matrix $\boldsymbol{G}_1$ and vector $\bar{\boldsymbol{f}}_1$ can be calculated as usual from recursive Diophantine equations (Clarke *et al.*, 1987). Alternatively, $\boldsymbol{G}_1$ is given from

$$\frac{B^-}{FA^-} = g_0 + g_1 z^{-1} + \cdots + g_{N+m} z^{-(N+m)} + \cdots \tag{26}$$

and $\bar{\boldsymbol{f}}_1$ can be calculated as the unstable system response from given initial conditions considering $\bar{u}(t+i) = 0, i \geq 0$.

The sequence of internal future control increments $\bar{\boldsymbol{u}}$ is then obtained from equality constraints (17) and yields (19). The matrix $\boldsymbol{G}_1$ can be shown to be always invertible. The filtered reference sequence has to correspond to the filtered output from (13) and must be constant. This yields (24).

Finally, the choice of the closed loop poles is respected by the filtration of the internal signals by $1/M$ and $\tilde{u}, y$ are given by (12), (13). $\square$

Table 1. Stabilising predictive strategies and the closed-loop poles

Authors	$G_1 \times G_2 \times G_3$	M
(Clarke and Scattolini, 1991; Mosca and Zhang, 1992)	$1 \times \frac{B}{AF} \times 1$	1
(Rossiter and Kouvaritakis, 1993)	$\frac{1}{AF} \times 1 \times B$	1
(Rawlings and Muske, 1993)	$1 \times \frac{B}{FA^-} \times \frac{1}{A^+}$	A^+
(Rossiter *et al.*, 1996)	$\frac{B^+}{FA^-} \times 1 \times \frac{B^-}{A^+}$	$(AB)^+$
(Fikar and Unbehauen, 2000)	$\frac{B^+}{A^+} \times \frac{B^-}{FA^-} \times 1$	B^+
This approach	$\frac{M}{A^+} \times \frac{B^-}{FA^-} \times \frac{B^+}{M}$	M

3.1 *Relation to other stabilising predictive strategies*

A comparison of stable predictive control methods can be performed for the same conditions. One useful way is to consider all methods without any degrees of freedom. The incorporation of the cost function and its minimisation changes only the properties of such nominal controller.

We have the following results: The proposed controller can be turned into any other stable predictive controller given in Table 1 by a suitable choice of the closed-loop poles M shown in Table 1 and given in the next theorem.

Theorem 4. Consider the scheme in Fig. 2 where

$$G_1 = \frac{G_{1N}}{G_{1D}}, \quad G_2 = \frac{G_{2N}}{G_{2D}}, \quad G_3 = \frac{G_{3N}}{G_{3D}}. \tag{27}$$

Assume that $\gcd(G_{1N}, G_{3D}) = M_0$, thus

$$G_{1N} = M_0\bar{G}_{1N}, G_{3D} = M_0\bar{G}_{3D}. \tag{28}$$

Then any predictive controller with no degrees of freedom and constraints (17), (18) is equivalent to the pole placement controller with poles

$$M = M_0\bar{G}_{1N}\bar{G}_{3D}. \tag{29}$$

The predictive controller is stable if and only if M is stable.

PROOF. The predictive controller is stable if $y, \tilde{u}$ are stable sequences (assuming regulation only for simplicity). The predictions are given as

$$y = \frac{G_{3N}}{G_{3D}}\bar{y}, \quad \tilde{u} = \frac{G_{1D}}{G_{1N}}\bar{u}. \tag{30}$$

Stability is then assured if G_{1N}, G_{3D} are both stable and if $\bar{y}, \bar{u}$ are stable sequences. The second requirement is automatically satisfied as $\bar{y}, \bar{u}$ are polynomials by assumption.

To derive the closed-loop poles we transform the predictions to have the same denominator

$$y = \frac{G_{3N}\bar{G}_{1N}M_0}{G_{3D}\bar{G}_{1N}M_0}\bar{y} = \frac{G_{3N}\bar{G}_{1N}}{M}\bar{y}, \tag{31}$$

$$\tilde{u} = \frac{G_{1D}\bar{G}_{3D}M_0}{G_{1N}\bar{G}_{3D}M_0}\bar{u} = \frac{G_{1D}\bar{G}_{3D}}{M}\bar{u}. \tag{32}$$

In order to arrive to the scheme in equation (11), the transfer function G_2 is fixed as $G_2 = B^-/FA^-$. The transfer functions are constrained by the relation

$$\frac{B}{FA} = \frac{M}{G_{1D}\bar{G}_{3D}}\frac{B^-}{FA^-}\frac{G_{3N}\bar{G}_{1N}}{M} \tag{33}$$

or

$$\frac{B+}{A+} = \frac{G_{3N}\bar{G}_{1N}}{G_{1D}\bar{G}_{3D}}. \tag{34}$$

Taking into account possible common factor T in this fraction it finally yields

$$G_1 = \frac{M}{TA^+}, \quad G_3 = \frac{TB^+}{M}. \tag{35}$$

Conversely, one may start from a stable M and derive the the condition that predictions are stable because G_{1N}, G_{3D} are stable. □

4. COST MINIMISATION

The natural way of obtaining the necessary degrees of freedom to minimise a cost function and to handle constraints is an assumption that the controller can act in more than N_u steps given by (14).

Let us define the following vectors and matrices

$$\bar{\boldsymbol{e}} = \bar{\boldsymbol{w}} - \bar{\boldsymbol{y}}, \tag{36}$$

$$\bar{\boldsymbol{y}}^T = (\bar{y}_{t+1}\ \bar{y}_{t+2}\ \dots\ \bar{y}_{t+N-1}), \tag{37}$$

$$\bar{\boldsymbol{f}}^T = (\bar{f}_{t+1}\ \bar{f}_{t+2}\ \dots\ \bar{f}_{t+N-1}), \tag{38}$$

$$\bar{\boldsymbol{w}}^T = (\bar{w}_{t+1}\ \bar{w}_{t+2}\ \dots\ \bar{w}_{t+N-1}), \tag{39}$$

$$\boldsymbol{G} = \begin{pmatrix} g_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ g_{N-1} & \dots & g_1 \end{pmatrix}, \tag{40}$$

with

$$\bar{\boldsymbol{y}} = \boldsymbol{G}\bar{\boldsymbol{u}} + \bar{\boldsymbol{f}}. \tag{41}$$

Theorem 5. Consider the predictive controller described in the Theorem 3. Introduce the number of degrees of freedom $n > 0$ and define the horizons

$$\bar{N}_u = N_u + n,\ \bar{N} = N + n. \tag{42}$$

Let us minimise the following cost function with $\boldsymbol{W}_e > 0$ and/or $\boldsymbol{W}_u > 0$

$$J = \bar{\boldsymbol{e}}^T\boldsymbol{W}_e\bar{\boldsymbol{e}} + \bar{\boldsymbol{u}}^T\boldsymbol{W}_u\bar{\boldsymbol{u}} \tag{43}$$

subject to the equality constraints (17), (18) and possible inequality constraints

$$\boldsymbol{A}\bar{\boldsymbol{u}} \geq \boldsymbol{b}. \tag{44}$$

If the optimisation problem is feasible then the predictive controller is stabilising for any n.

PROOF. Provided the signals $\bar{u}, \bar{e}$ are polynomials, the cost function J forms the standard Lyapunov function for which stability proofs are available. □

Let us now consider the "nominal" predictive controller, e.g. the task is to construct a controller with $n > 0$ such that in the unconstrained case its control actions are those of the nominal controller with $n = 0$ if the future setpoint is constant.

It can be shown (Fikar and Unbehauen, 1999) that such a controller is equivalent to the choice of the weights

$$\boldsymbol{W}_e = \langle \boldsymbol{X}_e \rangle, \boldsymbol{W}_u = \boldsymbol{0}, \tag{45}$$

where $\boldsymbol{X}_e$ is given by the solution of the spectral factorisation

$$B^* B = M_1^* M_1, \tag{46}$$

and the Diophantine equation

$$\frac{B_+^* B_+}{M_1^* M_1} \boldsymbol{P}_{N+n}^* \boldsymbol{P}_{N+n} = \frac{\boldsymbol{X}_e}{M_1} + \frac{\boldsymbol{Y}_e^*}{M_1^*}, \ \langle \boldsymbol{Y}_e^* \rangle = \boldsymbol{0}, \tag{47}$$

where

$$\boldsymbol{P}_{N+n}(z^{-1}) = (1 \ z^{-1} \ \ldots \ z^{-(N+n)}). \tag{48}$$

5. DISCUSSION

In this section we will show some simulation results related to the topics presented. Control of the following discrete system is considered

$$G = \frac{B^+ B^-}{A^+ A^-} = \frac{[(1 + 0.25z^{-1})][z^{-1}(1 + 4z^{-1})]}{\left[(1 + 0.2z^{-1})^2\right](1 + 5z^{-1})}. \tag{49}$$

In the first simulation we show the effect of the three optimal pole locations in pole placement control design as specified in Theorem 2. The results given in Fig. 3 can equivalently be obtained by the proposed predictive controller. In the latter case arbitrary $n > 0$ can be used with the "nominal" predictive controller and assuming that the future setpoint is constant.

If the future setpoint is known, the actions of the pole placement and predictive controller are no longer the same. Consider for example the case $M = B^+$ (control error dead-beat) shown in Fig. 4. The solid line corresponds to the previous simulation, the dotted line to the predictive controller with $n = 8$ degrees of freedom. The predictive controller starts its actions before the actual setpoint change and thus obtains faster output response. Note, that also in this case the requirement of the control error dead-beat is satisfied even if the cost function is minimised.

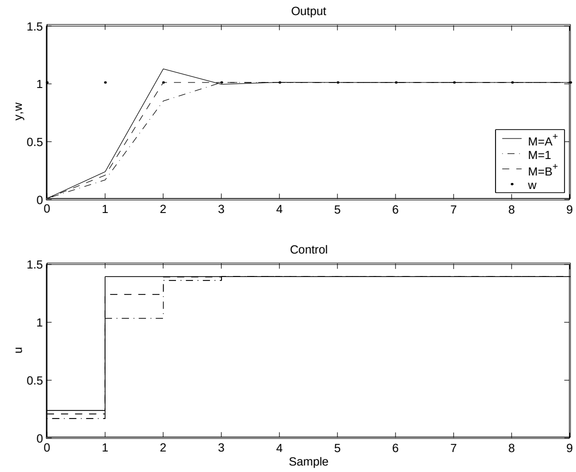

Fig. 3. Effect of the closed-loop poles, time optimal control

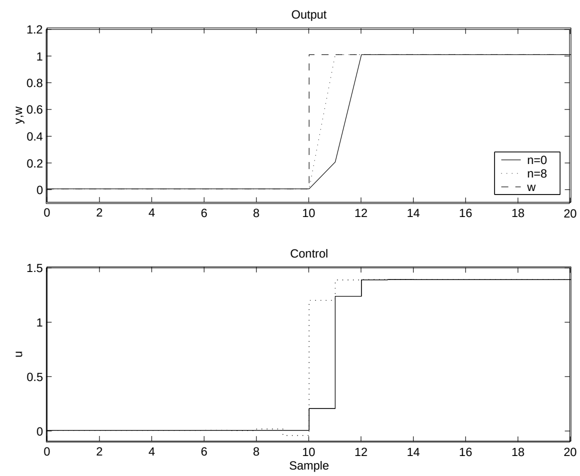

Fig. 4. Effect of the knowledge about the future setpoint

6. CONCLUSIONS

This paper has discussed some relations between pole placement and predictive control design.

From the PP point of view, the results obtained for three typical time optimal control strategies can be linked to the minimum control and output horizons for stable predictive control. To be more specific, the minimum control horizon has to be greater or equal to the number of unstable poles of the system and the reference, and the minimum output horizon has to be greater or equal to the number of unstable zeros of the system. Although some of these results have been discovered in MPC, here they are obtained from the closed-loop assumptions.

From the MPC point of view, a general formulation has been proposed that links some stable MPC strategies together via the choice of the closed-loop polynomial M. Again, some of the results have been known before, here they are obtained from different assumptions.

The choice of the cost function, that produces the nominal PP controller in unconstrained case has been given. This holds if assuming unknown future setpoint sequence.

Finally some simulations have been presented to illustrate the ideas of the paper.

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