MULTIRATE APPROACH TO NONLINEAR PREDICTIVE CONTROL

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Abstract: The article discusses a possible approach to the reduction of the computational load of a nonlinear model predictive controller. In principle, the sufficiently large output horizon is divided into only a few equidistant intervals with piece-wise constant control actions. After finding a solution of this dynamic optimisation problem, the control horizon is halved keeping parts of the first solution fixed, the sampling rate is doubled and the whole procedure is repeated until the length of the first time interval is reasonable. This procedure is repeated using the receding horizon principle.

Keywords: nonlinear systems, predictive control

1. INTRODUCTION

The success of model predictive control (MPC) framework is based on its relatively simple idea as well as on its practical properties, namely its ability to cope with constraints and nonlinearities. In principle, in each sampling interval, a finite horizon optimisation problem is solved and the first element of the control trajectory is applied to the controlled process.

Nowadays, properties of MPC are well understood in the linear case. However, less results are available for the nonlinear case. Therefore, this paper is focused onto the nonlinear case. Also here, stability properties can be proven by the use of different strategies. A survey can be found for example in (De Nicolao *et al.* 2000) showing concepts of zero-state terminal constraint (Keerthi and Gilbert 1988), dual mode controller (Michalska and Mayne 1993), quasi-infinite controller (Chen and Allgöwer 1998), etc.

However, the main obstacle in practical applications of nonlinear MPC (NMPC) is not stability, but the computational burden. A stable predictive controller has to guarantee, that the origin or some defined region is reached at the end of the horizon. Feasibility of this type of problems increases with horizon length. On the other hand, the horizon length increases the computational efforts exponentially.

There are only a few methods that address the "curse of dimensionality". Most often, some part of the problem is treated as linear. The approaches include the use of a bank of linear models (Foss *et al.* 1995), linearisation at the current operation point (Banerjee and Arkun 1998), or optimisation of only the actual control move and supposing that all other future moves are calculated by a saturated linear controller (Zheng 2000).

The aim of this contribution is to present a completely different approach to reduce the computational complexity of NMPC. The developed new approach assumes a nonlinear model in all calculation steps. The reduction of the computational load is achieved by sampling the control trajectory in a non-equidistant way, placing the shortest sampling intervals at the beginning of the horizon and increasing the following intervals exponentially with time.

2. PROBLEM SETUP

Consider a time-invariant nonlinear continuous-time system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{1}$$

where $\boldsymbol{x}(t) \in R^x$ is the state, $\boldsymbol{u}(t) \in R^u$ is the input, $\boldsymbol{f}$ is a nonlinear function satisfying $\boldsymbol{f}(\boldsymbol{x}^s, \boldsymbol{u}^s) = \boldsymbol{0}$. $\boldsymbol{x}^s, \boldsymbol{u}^s$ are the desired states and the hereto corresponding control. The state and input vectors are subject to the constraints

$$\boldsymbol{x}(t) \in X, \quad \boldsymbol{u}(t) \in U, \quad t \geq 0, \tag{2}$$

where X and U are compact sets of R^x and R^u respectively, with $\boldsymbol{x}^s \in R^x$ and $\boldsymbol{u}^s \in R^u$.

The cost function at sampling time k is defined as follows:

$$J = \int_0^T \left(||\boldsymbol{x}^s - \boldsymbol{x}(t|k)||^2_{\boldsymbol{W}_x} + ||\boldsymbol{u}^s - \boldsymbol{u}(t)||^2_{\boldsymbol{W}_u} \right) dt, \tag{3}$$

where T is the prediction horizon and $\boldsymbol{W}_u > 0, \boldsymbol{W}_x > 0$ are weighting matrices. Piece-wise constant control steps are applied, whereas states are continuous.

Assumption 1 (feasibility): For a given $T > 0$ and a number of control steps $n > 0$, there exists a nonempty neighbourhood $X(T)$ of the state $\boldsymbol{x}^s$ such that $\forall \boldsymbol{x}(0) \in X(T)$ there exists a control sequence $\boldsymbol{u}(k)$, $k = 1, \ldots, n$ that drives the state of (1) to $\boldsymbol{x}^s$, i.e. $\boldsymbol{x}(T) = \boldsymbol{x}^s$, and such that the constraints (2) are satisfied on the interval $[0, T]$.

3. DESCRIPTION OF THE METHOD

One of the main issues in NMPC is that of computational load. If stability is to be assured, a long horizon T has to be chosen which corresponds to a large number of optimised control variables. On the other side, only the first element of the calculated control trajectory is used and all others are discarded even if they were calculated. This suggest the idea to sample the optimised control trajectory exponentially.

We consider the number of optimised control segments to be equal to n and specify the horizon T such that the feasibility assumption is satisfied. A possible approach is then to divide the control trajectory into $m \geq n$ exponentially growing sampling intervals $\Delta_1, \Delta_2 = 2\Delta_1, \ldots, \Delta_m = 2^{m-1}\Delta_1$. Using this pattern for the predicted control trajectory of the optimisation will lower the calculation costs considerably, compared with the classical methods using equidistant intervals. However, in this contribution, a different recursive multirate approach has been adopted which simplifies the optimisation task furthermore:

Consider the desired end time $t_e = T$. Set the iteration counter $i = 1$ and the desired number of the recursions n_r. The approach is then based on the following steps:

(1) Divide the time interval $[0, t_e]$ into an even number m of sampled time intervals, each of length $\Delta = t_e/m$, with piece-wise constant control actions and optimise the cost function

$$\begin{aligned} &\min_{\boldsymbol{u}[0,t_e]} J^i = \int_0^T \left(||\boldsymbol{x}^s - \boldsymbol{x}(t|k)||^2_{\boldsymbol{W}_x} \right. \\ &\qquad \left. + ||\boldsymbol{u}^s - \boldsymbol{u}(t)||^2_{\boldsymbol{W}_u} \right) dt \\ &\text{subject to} \\ &\boldsymbol{u}(t_e, T] = \min J^{i-1} \\ &\boldsymbol{x}(T) = \boldsymbol{x}^s \\ &\boldsymbol{x} \in X, \quad \boldsymbol{u} \in U, \end{aligned} \tag{4}$$

where the notation $\boldsymbol{u}(t_e, T] = \min J^{i-1}$ corresponds to the optimal control trajectory in the time interval $(t_e, T]$ from the previous iteration.

(2) Define the new endpoint as $t_e := t_e/2$ (in the middle).

(3) Increase the iteration counter by one and go to Step 1 if $i < n_r$.

The main features of this procedure are:

- The optimisation problem is solved n_r times with only m control variables.
- The smallest sampling interval used in the predictive control settings at time $t = 0$ is $\Delta_1 = 2^{-n_r+1}T/m$.

The principle is illustrated in Fig. 1 where it was assumed that two control segments are optimised in each recursion and the first three recursions are shown ($m = 2$, $n_r = 3$, $T = 8$). The smallest sampling interval is thus $\Delta_1 = 1$. For comparison, the classical equidistantly sampled approach to NMPC would require one optimisation with 8 optimised variables solved simultaneously. The difference would be more striking if the number of recursions is increased.

Clearly, the exponential growth of the sampling interval length makes it easier to define an overall horizon T large enough to satisfy the feasibility (and thus also stability) assumption.

The main differences between the direct choice of the exponentially growing sampling intervals and the proposed approach lies in the number of optimised variables and the complexity of the optimisation task. With the proposed approach, the number of optimised variables in each recursive step is less than for one static optimisa-

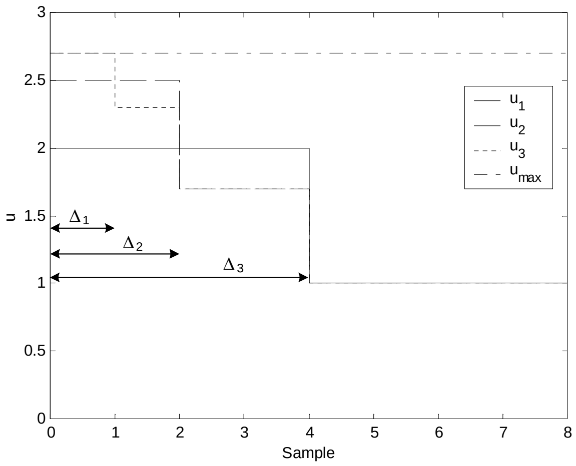

Fig. 1. Principle of the multirate approach

tion problem and therefore, better convergence properties can be expected. Moreover, using small intervals at the beginning of the control trajectory inevitably leads to excessive control actions, violation of the constraints, and represents a more difficult task for the optimiser. On the other hand, the proposed multirate approach requires a recursive series of optimisation problems to be solved. As in the first recursion large time intervals occur, a feasible solution can easily be found which is more unlikely to violate the constraints. Initialisation of the subsequent recursive problems is straightforward due to the knowledge of the solution from the preceding recursion, which is also a possible solution of all subsequent problems. This, the so called *feasible path* iterative procedure enables at any time either to stop the optimisation and use the actual solution or to use the available computational time to improve the predicted control strategy furthermore. The results of any recursive step represents a control strategy that satisfies the terminal state constraint condition and will therefore also guarantee stability.

The practical implementation of this procedure has been solved using the orthogonal collocation technique on finite elements (Villadsen and Michelsen 1978). The state trajectory has been approximated by the orthogonal Lagrange polynomials piecewise on the intervals corresponding to the control intervals.

The polynomial approximations for state variables at one interval can be expressed as

$$x(t) = \sum_{j=1}^{N} \bar{x}_j \phi_j(t), \quad \phi_j(t) = \prod_{i=1,j}^{N} \frac{t - t_i}{t_j - t_i}, \quad (5)$$

and $N - 1$ is the degree of the Lagrange polynomial. The notation $i = 1, j$ denotes i starting from one but excluding $i \neq j$. The times t_i are given as the roots of the Legendre polynomials (Villadsen and Michelsen 1978). It is worth noticing that

$$x(t_i) = \bar{x}_i \quad (6)$$

and thus the unknown coefficients $\bar{x}_j$ are physically meaningful quantities. This becomes useful when initialising the state variable profile.

4. SIMULATION RESULTS

The properties of the procedure described in the previous section are studied by means of simulations. As a controlled system we consider a double integrator described by the state-space representation

$$\dot{x}_1 = u, \quad x_1(0) = 1, \quad (7)$$

$$\dot{x}_2 = x_1, \quad x_2(0) = 0, \quad (8)$$

which is to be steered to the state $[-1, 0]$ with the constraint on minimum value of the control variable $u_{min} = -0.5$ and the cost function

$$J = \int_0^T [(x_1 + 1)^2 + x_2^2 + \lambda u^2] dt. \quad (9)$$

Unless otherwise stated, the number of optimised control segments was set to $m = 4$. The first simulation shown in Fig. 2 illustrates properties of the algorithm for several values of number of recursions n_r. The output horizon and the weighting coefficient were set to $T = 16$s and $\lambda = 8$ respectively. If only one recursion is allowed, the sampling time for the predictive control is $T/m = 4$s. One can notice that in this case the control does not hit the lower constraint. Increasing n_r quickly leads to an asymptotic continuous-time control trajectory. The sampling time with $n_r = 6$ is $\Delta_1 = 0.125$s and the corresponding optimisation problem with 4 variables has to be solved 6 times per sampling time.

In the second simulation (Fig. 3), the results using different values of the weighting coefficient are shown for $T = 8$s and $n_r = 4$. The variation of the penalisation coefficient has the desired effect of influencing the speed of the controller and can be considered as the tuning knob of the method.

Finally, the last simulation (Fig. 4) compares the proposed predictive controller with a classical predictive controller using the same sampling time 0.5s, output horizon $T = 8$s and weighting coefficient $\lambda = 1$. We can see only a small difference in the control trajectories. However, the comparison of the number of optimised variables shows 16 variables for the classical NMPC case against 3 recursions with 4 optimised variables using the proposed approach. The simulation showed that approximately 30 times more floating point operations were needed for the classical case compared with the multirate approach, although only 3 recursion steps were calculated. This difference in speed will grow rapidly with increased number of recursions.

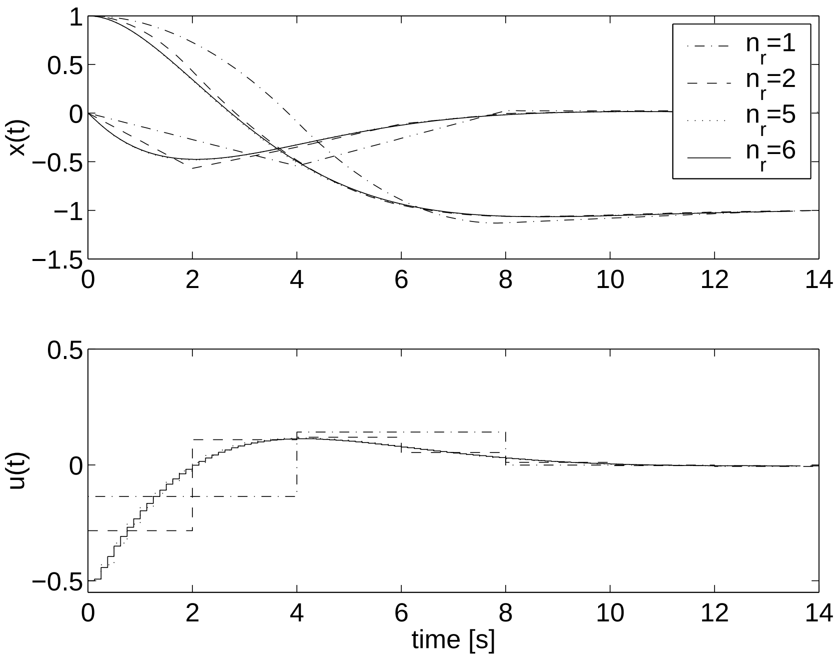

Fig. 2. Different number of recursions n_r

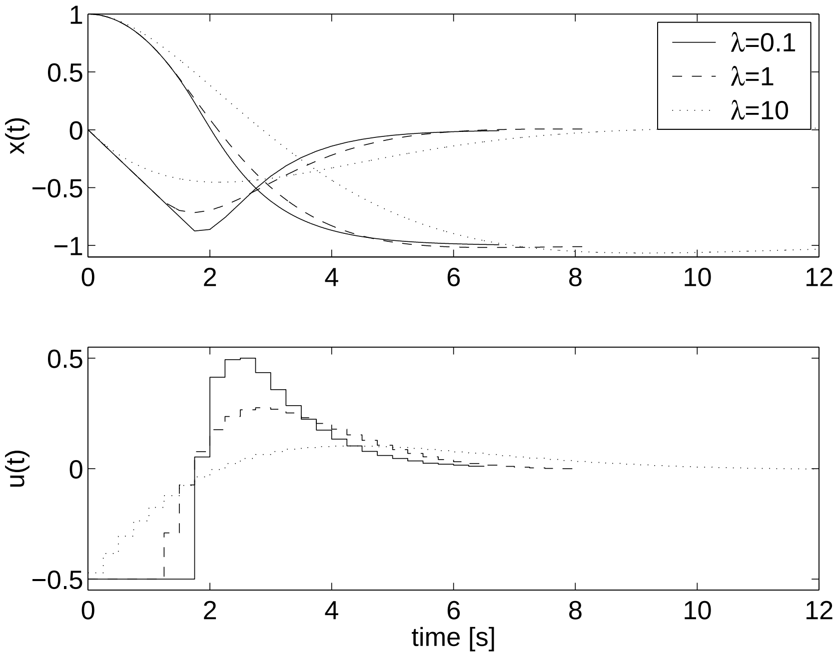

Fig. 3. Different values of λ

5. CONCLUSIONS

This contribution has proposed a new multirate recursive approach for the design of a new nonlinear model predictive controller. The aim was to reduce the computational effort needed in the classical approach, without violating stability conditions. The main idea is to give the control trajectory the most precision at the present time since the quality of prediction will decay with increased horizon lengths. The assumption was made that the control trajectory can be approximated with exponentially increasing sampling lengths. This approximation approach can be justified since only the first part of the future control trajectory is used for control and the rest is discarded.

To calculate the control trajectory, a multirate recursive approach has been proposed. This consists of optimising at each recursion m control actions equidistantly sampled, but halving the output horizon in each recursion.

At present, the guarantee of stabilising feature of the procedure is based on the terminal constraint approach. Future research is focused onto gener-

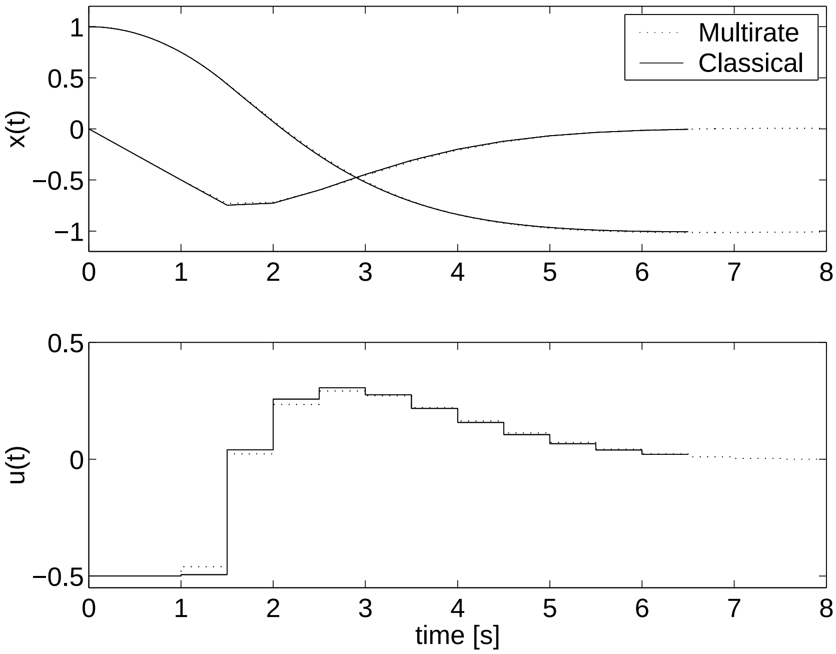

Fig. 4. Comparison with a standard predictive controller

alising this idea to terminal regions and thus enlarging the feasibility properties of the algorithm.

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6. REFERENCES

Allgöwer, F. and Zheng, A., Eds.) (2000). *Nonlinear Model Predictive Control*. Vol. 26 of *Progress in Systems and Control Theory*. Birkhäuser. Basel.

Banerjee, A. and Y. Arkun (1998). Model predictive control of plant transitions using a new identification technique for interpolating nonlinear models. *Journal of Process Control* **8**(5–6), 441 – 457.

Chen, H. and F. Allgöwer (1998). A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica* **34**(10), 1205 – 1217.

De Nicolao, G., L. Magni and R. Scattolini (2000). Stability and robustness of nonlinear receding horizon control. pp. 3 – 22. In: Allgöwer and Zheng (2000).

Foss, B. A., T. A. Johansen and A. V. Sorensen (1995). Nonlinear predictive control using local models applied to a batch fermentation process. *Control Engineering Practice* **3**(3), 389 – 396.

Keerthi, S. S. and E. G. Gilbert (1988). Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: stability and moving horizon approximations. *Journal of Optimization Theory and Application* **57**, 265 – 293.

Michalska, H. and D. Q. Mayne (1993). Robust receding horizon control of constrained nonlinear systems. *IEEE Tr. Aut. Control* **38**, 1623 – 1633.

Villadsen, J. V. and M. L. Michelsen (1978). *Solution of Differential Equation Models by Polynomial Approximation*. Prentice Hall, Englewood Cliffs, NJ.

Zheng, A. (2000). Some practical issues and possible solutions for nonlinear model predictive control. pp. 129 – 143. In: Allgöwer and Zheng (2000).