MULTI-RATE APPROACH TO PREDICTIVE CONTROL

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Abstract: Nonlinear model predictive control (NMPC) based on nonequidistant sampling rate has been proposed (Halldorsson, 2002; Halldorsson et al., 2004). This implementation can achieve significant computational savings compared to classical methods without noticeable deterioration of the control quality. On the other side, classical stability results cannot be aplicable as the standard MPC feasibility assumption can be broken.

Keywords: nonlinear systems, predictive control, multirate control

1. INTRODUCTION

Receding-Horizon (RH) control, also known as Model Predictive Control (MPC), can in general be formulated as a repeated solution of an open-loop optimal control, where the dynamics and constraints of the underlying system are taken into account as optimization boundary conditions. This simple idea behind MPC, as well as its practical properties, make it easy to implement system nonlinearities and constraints directly into the control law. Optimal control trajectory over some finite horizon is approximated as a piece-wise constant sequence of which the first part is applied to controlled process.

Nonlinear model predictive control (NMPC) follows the success of linear MPC. Various strategies have been developed that guarantee stability of the closed-loop system, see for example Allgöwer et al. (1999); Morari and Lee (1999); De Nicolao et al. (2000); Mayne et al. (2000).

Besides stability, another major issue with NMPC is its computational burden. To lower the complexity of optimization problems it is desirable to apply a small number of optimizable variables. When considering nonlinear models it is straightforward to distinguish between the prediction horizon and the number of optimised variables. Clearly, the best performance is attainable if these two coincide, but this will take place at the cost of excessive computational burden. Various approaches have been suggested at this point, mostly based on selecting a low number of optimised variables. In Bemporad (1998) a reference governor is implemented which searches for a future setpoint trajectory parameterised by a single optimised parameter. Another approach inspired by linear MPC maintains constant predicted control actions beyond a certain point in the control horizon, or uses as a reference the output of a suitable linear controller beyond that point. Following this scheme Zheng (2000) and Magni et al. (2001) optimise the nonlinear control actions only m steps into the future, while assuming the remaining future control strategy is generated by a linear controller. For this case some nice convergence features have been proven for arbitrary m.

This contribution studies reduction of computations by nonequidistant parametrisation of control steps as described in Halldorsson (2002); Halldorsson et al. (2004). The sampling frequency is high at the beginning of the optimal control trajectory and decreases exponentially towards the optimisation horizon. This type of parametrisation lowers computational load significantly without noticeable loss of performance.

However, this special parametrisation does not obey the standard properties for feasibility argument needed to prove stability by the existing approaches.

We focus in this contribution on both properties and discuss possible ways for guaranteing stability.

2. STANDARD PROBLEM SETUP

We consider a standard NMPC problem with a nonlinear continuous-time system with control applied at discrete sampling instants.

The time-invariant nonlinear continuous-time system is given as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{1}$$

where $\boldsymbol{x}(t) \in R^{\mathrm{x}}$ is the state vector, $\boldsymbol{u}(t) \in R^{\mathrm{u}}$ is the input vector, $\boldsymbol{f}$ is a twice differentiable nonlinear function locally Lipschitz continuous satisfying $\boldsymbol{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. The state and input vectors are subject to the constraints

$$\boldsymbol{x}(t) \in X, \quad \boldsymbol{u}(t) \in U, \quad t \geq 0, \tag{2}$$

where X and U are compact sets of R^{x} and R^{u} respectively, both enclosing the origin representing the steady-state of the system. Considering (1) and (2) as boundary constraints a general objective function to be minimised is defined as

$$J(\boldsymbol{x}(t), \boldsymbol{u}(\cdot)) = G[\boldsymbol{x}(t+T)] + \int_t^{t+T} F[\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)] \mathrm{d}\tau, \tag{3}$$

where T is the prediction horizon length, and with slight abuse of notation $\boldsymbol{x}(\tau)$, $\boldsymbol{u}(\tau)$ for $\tau \in [t, t+T]$ are the future predicted values of $\boldsymbol{x}(\tau)$ and $\boldsymbol{u}(\tau)$, respectively. The term $\boldsymbol{u}(\cdot)$ denotes the input trajectory ofver the horizon. $F > 0$, $G \geq 0$ are general functions describing the desired objective and G serves as the terminal penalty term. For stability reasons it may be required to augment this optimisation problem with the constraint $\boldsymbol{x}(t+T) \in \Omega$, where Ω is a compact set defined according to the applied stability proof scheme.

Stability of the closed-loop system can be established based on feasibility and decrease of the value function (the minimal value of the cost for the state $\boldsymbol{x}$: $V(\boldsymbol{x}) = J(\boldsymbol{u}^*, \boldsymbol{x})$) as follows:

Theorem 1. (Findeisen et al. (2003)). *Suppose that*

- *the terminal region $\Omega \in X$ is closed with $0 \in \Omega$ and that $G > 0$*
- *$\forall \boldsymbol{x} \in \Omega$ there exists an admissible input $\boldsymbol{u}_\Omega(\tau)$ such that $\boldsymbol{x}(\tau) \in \Omega$ and*

$$\frac{\partial G}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}(\tau), \boldsymbol{u}_\Omega(\tau)) + F(\boldsymbol{x}(\tau), \boldsymbol{u}_\Omega(\tau)) \leq 0 \quad \forall \tau \in [0, T] \tag{4}$$

- *the NMPC open-loop optimal control problem is feasible for $t = 0$.*

Then the closed-loop system is asymptotically stable and the region of attraction $\mathcal{R}$ consists of the states for which an admissible input exists.

For the purposes of our presentation, we will use the arguments for robustness of sampled-data NMPC as given in Findeisen et al. (2003). To do so, we will assume that:

Assumption 1. A perturbed controlled system with piece-wise continuous bounded $\boldsymbol{v}$ describing the input uncertainty

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u} + \boldsymbol{v}) \tag{5}$$

has a continuous solution for any $\boldsymbol{x}(0) \in \mathcal{R}$, any piece-wise continuous input $\boldsymbol{u}$ and $\boldsymbol{v}$.

Assumption 2. The value function $V(\boldsymbol{x})$ is continuous

Assumption 3. There exists a strictly increasing function α_v with $\alpha_v(0) = 0$ such that for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{R}$: $V(\boldsymbol{x}_1) - V(\boldsymbol{x}_2) \leq \alpha_v(||\boldsymbol{x}_1 - \boldsymbol{x}_2||)$

We now define a level set Ω_c of V in $\mathcal{R}$ where $c > 0$ and $\Omega_c = \{\boldsymbol{x} \in \mathcal{R} | V(\boldsymbol{x}) \leq c\}$. Then we assume that

Assumption 4. For all compact sets $\mathcal{S} \subset \mathcal{R}$ there is at least one level set Ω_c such that $\mathcal{S} \subset \Omega_c$.

The level sets help to define a weaker notion of stability than asymptotic stability as usually guaranteed by NMPC. Here, we desire a bounded stability; that the norm of the state after some time becomes small. The results are based on the observation that small uncertainties in control lead to a small difference between the predicted and real states.

The above assumptions are standard as defined in Findeisen et al. (2003). Based on them, the following fact is needed:

Fact 1. For any $c > \alpha > 0$ with $\Omega_c \subset \mathcal{R}$, $T > \delta > 0$ the lower bound $V_{\min}(c, \alpha, \delta)$ on the value function exists and is non-trivial for all $\boldsymbol{x}_0 \in \Omega_c / \Omega_\alpha$

$$0 < V_{\min}(c, \alpha, \delta) = \min_{\boldsymbol{x}_0 \in \Omega_c / \Omega_\alpha} \int_o^\delta F(\boldsymbol{x}(\tau, \boldsymbol{x}_0), \boldsymbol{u}^*(\tau, \boldsymbol{x}_0) d\tau < \infty \quad (6)$$

3. MULTIRATE OPTIMAL CONTROL

To solve the control problem, a future control trajectory needs to be calculated at each sampling time and its first part is applied to the process. The usual assumption regarding this trajectory is that the control actions can be parametrised as piece-wise constant. Hence, the original dynamic problem can be regarded as a problem of static optimisation (nonlinear programming – NLP) with control action values.

The complexity of the optimisation depends on the prediction horizon – final time T and on the number of degrees of freedom (piece-wise constant control moves). In order for the predictive control to take the nonlinear nature of the process fully into account, a sufficiently long prediction horizon must be chosen, enclosing the transients of the operating point change. Feasibility and quality of control can be improved by increasing the prediction horizon length T. This leads however, for the classical case, to a large number of optimised control variables and a tedious optimisation problem to be solved in one step.

As it has been proposed in Halldorsson (2002), it is possible to optimise with control steps of exponentially increasing time intervals $T_s, 2T_s, 4T_s, \ldots, T/2$. The main advantages of the method are:

- As only the first element of the optimal solution will be applied to the process, it is obvious that the latter part od the trajectory needs not the be very precise.
- Due to possible uncertainties and disturbances, accuracy of state and control predictions decreases with increasing prediction horizon.
- The requirement of approximate steady-state at the end of the horizon dictates the last part of the optimal trajectory approaching zero for a sufficiently long time – this is implicitly considered here.

Several other advantages of the method can be found if the optimal control trajectory is found in multiple recursive optimisations with a low number of degrees of freedom. However, for the purpose of this contribution, the actual implementation is not important.

The main drawback of the multirate method is theoretical and follows from the fact that such a parameterisation cannot in general guarantee feasibility at $t > 0$ as the optimal control trajectory at $t = 0$ cannot be used in the next sampling time.

In order to cope with the multirate approach, we assume additionally that

Assumption 5. The sampling period T_s of the multirate NMPC is sufficiently small.

Based on this assumption, it can be concluded that the optimal control trajectory calculated at time t is also feasible as an perturbed control trajectory at $t + T_s$. Due to the lack of space we will only sketch the further steps.

It is possible to obtain

$$||\boldsymbol{x}(t_k + T_s) - \bar{\boldsymbol{x}}(t_k + T_s)|| \leq \int_{t_k}^{t_k+T_s} L_{fx} ||\boldsymbol{x}(s) - \bar{\boldsymbol{x}}(s)|| ds + L_{fu} \boldsymbol{v}_{\max} T_s \quad (7)$$

where L_{fu} is the Lipschitz constant of $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ with respect to $\boldsymbol{u}$, $\boldsymbol{v}_{\max}$ is the upper bound of $\boldsymbol{v}$, t_k represents start of a k-th sampling interval and the predicted variables are denoted by bar. Applying the Gronwall-Bellman inequality and from properties of the value function yields

$$||\boldsymbol{x}(t_k + T_s) - \bar{\boldsymbol{x}}(t_k + T_s)|| \leq \frac{L_{fu} \boldsymbol{v}_{\max}}{L_{fx}} (e^{L_{fx} T_s} - 1) \quad (8)$$

$$\alpha_v \left(\frac{L_{fu} \boldsymbol{v}_{\max}}{L_{fx}} (e^{L_{fx} T/2} - 1) \right) \leq c - c_0 \quad (9)$$

From this on we obtain the following result:

Theorem 2. Given the level sets $\Omega_\alpha \subset \Omega_{c0} \subset \Omega_c \subset \boldsymbol{R}$ and assuming that the error resulting from small time shift T_s satisfies $||\boldsymbol{u}|| \leq \boldsymbol{u}_{\max}$ and that

$$\alpha_v \left(\frac{L_{fu} \boldsymbol{v}_{\max}}{L_{fx}} (e^{L_{fx} T/2} - 1) \right) \leq \min \{ (c - c_0), V_{\min}(c, \alpha/4, T_s), \alpha/2 \} \quad (10)$$

Then for any $\boldsymbol{x}(0) \in \Omega_{c0}$ the closed-loop trajectories under the optimal feedback will not leave the set Ω_c and there exists a finite time T_α such that $\boldsymbol{x}(\tau) \in \Omega_\alpha \ \forall \tau \geq T_\alpha$.

Considering the assumptions, Assumption 2 is generally very difficult to guarantee. This concerns NMPC with state constraints in general and multirate procedure in particular.

The stability result given here is only of qualitative nature as the exact minimum bounds as in Fact 1 cannot easily be found.

Receding horizon control with arbitrary open-loop trajectories calculated in each sampling step can be thought as a generalisation of the multirate procesure. Clearly, it still remains a theoretical challenge to solve quantitatively stability of such scheme.

4. SIMULATION RESULTS

The aim of the simulations is to show some properties of the multirate scheme.

4.1 Decreasing Sampling Time

In the first part the effect of decreasing sampling time will be investigated. Consider a nonlinear system of the form

$$\begin{aligned} \dot{x}_1(t) &= [0.5 + \cos(x_2(t))]x_1(t) \\ &\quad + [0.5 + \sin(x_2(t))]u(t), \\ \dot{x}_2(t) &= x_1(t), \end{aligned} \tag{11}$$

with the input signal constraints

$$-1.0 \le u(t) \le 1.0 \quad \text{for} \quad t \ge 0, \tag{12}$$

subject to the initial state $\boldsymbol{x} = \boldsymbol{0}$ and the desired final state $\boldsymbol{x}_F = [0\ \pi]^T$. The objective function is defined as

$$J(\boldsymbol{x}(t), \boldsymbol{u}(\cdot)) = \int_t^{t+T} ||\boldsymbol{x}(\tau) - \boldsymbol{x}_F||^2 + ru(\tau)^2)\mathrm{d}\tau, \tag{13}$$

where the parameters are chosen as $r = 2$, $T = 6$. To assure a stable closed-loop behaviour the terminal equality constraint $\boldsymbol{x}(t+T) = \boldsymbol{x}_F$ will be applied.

Multirate control was applied with varying number N_r of optimisation recursions and with the basic sampling time $T_s = 1.5$. The case with $N_\mathrm{r} = 1$ corresponds to a standard NMPC with $m = 4$ equidistant sampling intervals of length T_s. If $N_\mathrm{r} = 2$ then there are first four sampling periods of length $T_s/2$ followed by two sampling periods with T_s. Further recursion double the number of sampling periods in the first half of partitions from the previous recursion.

Simulation results are shown in Figure 1. The figure shows that as the number N_r of suboptimisation steps is increased the results converge to an identical behaviour. As a matter of fact, deviation in the state variables $x_1(t)$ and $x_2(t)$

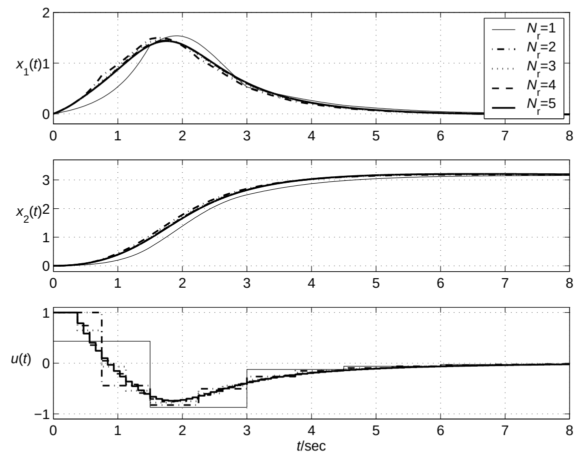

Fig. 1. Increasing the number of suboptimisation steps ($r = 2$)

from the expected results of convergence, say for $N_\mathrm{r} = 5$, is clearly detectable only for the first two settings $N_\mathrm{r} = 1$ and $N_\mathrm{r} = 2$. For $N_\mathrm{r} \ge 3$ deviations only appear in control $u(t)$, which originate from the different sampling rates applied. This confirms the theoretical results that decrease of the sampling time results in small deviations from optimal control.

4.2 Distillation Column Control

We consider a distillation column with 19 trays for separation of a binary mixture of methanol and iso-propanol Hallager et al. (1986). Its detailed model has been described in Fikar et al. (1998). In principle, every tray is described by two differential equations for overall molar (w_i) and methanol (x_i) molar balances, and by two algebraic equations for vapour-liquid relationship (y_i, x_i) and liquid molar flow (L_i).

The model of the column is considered as a multivariable system with two inputs and two outputs. The inputs $\boldsymbol{u}$ are the reflux flowrate and the reboiler duty (assumed proportional to the bottom vapour flowrate). In order to prevent weeping and flooding of the column, the reflux is constrained to be within lower and upper limits

$$0.186 \le R \le 0.996. \tag{14}$$

The molar vapour flowrate V is constrained by its mass balance equations. Practically, the constraints imposed on V are :

$$R + 0.01 \le V \le R + 0.06. \tag{15}$$

The output variables are liquid mole fractions of methanol in reboiler x_B and in condenser x_D.

The problem of reaching a new desired steady state has been formulated as the optimisation problem of minimising the integral square error (ISE)

$$J = \min_u \int_0^T (x_B(t)-x_B^s)^2+(x_D(t)-x_D^s)^2 dt \quad (16)$$

The desired final steady state was specified for molar fractions of methanol in distillate and bottom flows as $x_B^s = 0.04$, $x_D^s = 0.93$. Based on analysis of dynamical properties of the column, the final time was set to $T = 125$ min and to show the potentialities of the method, the control was divided into $m = 32$ piece-wise constant segments.

Different NMPC strategies were employed. In the first one, full $m = 32$ segment control trajectory was calculated, using only the first control segment for control, as is usually done in MPC. In the second one, the proposed approach was implemented with $m = 2$ optimised piece-wise constant controls and with $n_{\mathrm{r}} = 5$ recursions – that is, the optimisation is performed 5 times with 2 optimised control segments. These values ensure that the actually used sampling interval is the same as for the first strategy. The results are shown in Fig. 2 and only a minor deviation can be observed. Comparison of the actually attained cost at $T = 125$ min gives only a very minor deterioration of 0.01% of the proposed approach compared to the classical one. Also in this case, the differences are negligible. However, the comparison of the computational time reveals that the proposed algorithm is almost 10 times faster (157 s versus 1479 s per one NMPC step). We can conclude, that the proposed approach reduces the computational time significantly without any large compromise in the control quality.

Another quite popular approach to reduce the computational time is to utilise the argument of the control horizon N_u, after which the control variable remains constant. To make a fair comparison, $N_u = m = 2$ was set. Of course, the computational time is smaller for this case, as only one optimisation with 2 variables is performed (when compared with 5 optimisations with 2 controls in multirate case). However, the results indicate, that this optimisation is more difficult and takes approximately 104 s per one NMPC step.

The comparison of this strategy with the full approach is shown in Fig. 3. It can be seen that the trajectories are no longer very similar and it comes to deterioration of the control quality. Also the actually attained cost is about 0.90% higher than the full approach.

Compared with the proposed approach, both methods construct similar control trajectories in the first part (up to 40 min) where not much room for optimisation exists due to the constraints. However, the unconstrained part of the control trajectories differs significantly as the N_u approach realises an asymptotic type of the transients whereas the proposed method reacts more actively.

5. CONCLUSIONS

This contribution has dealt with nonlinear model predictive control where the open-loop control problem is characterised by an exponentially increasing sampling time instants. This gives cloed-loop optimal control and state trajectories very similar to the classical equidistant NMPC, however with only a fraction computational time.

Special type of control parameterisation causes theoretical problems of stability guarantee even if the original NMPC framework is stabilising. We have proposed here some stability properties of the method. However, the quantitative result is nowadays a very chalenging problem and the question remains how to guarantee stability without the usual feasibility assumption.

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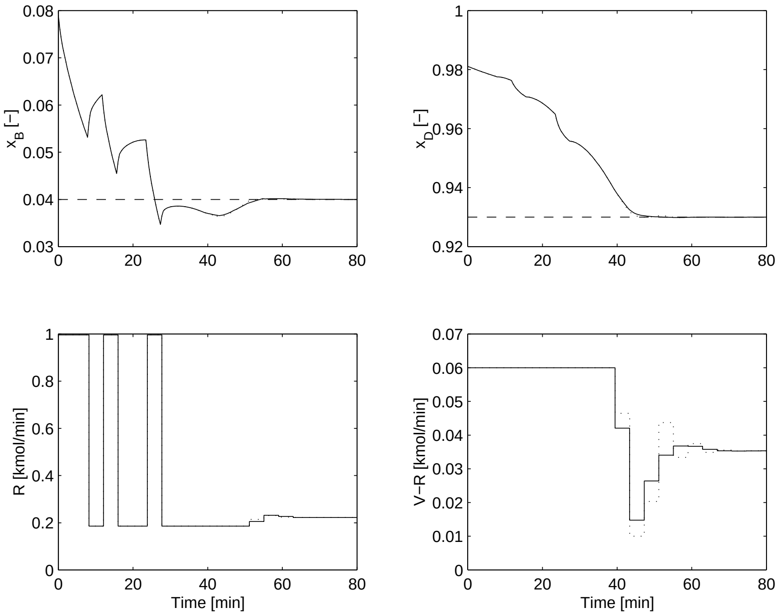

Fig. 2. Comparison of the multirate (full line) and standard (dotted line) approaches

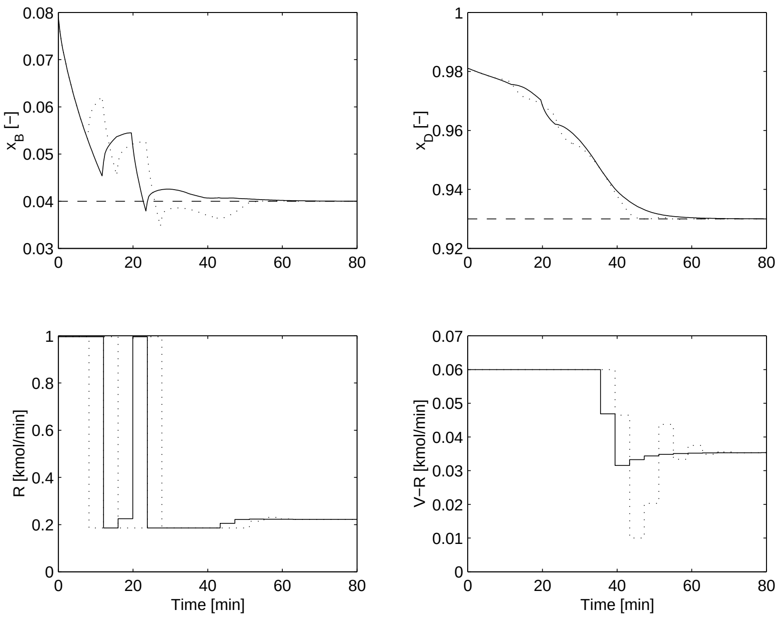

Fig. 3. Comparison of the reduced - N_u (full line) and standard (dotted line) approaches

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