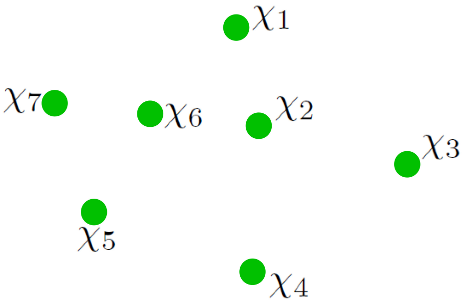
ALADIN—An Algorithm for Distributed Non-Convex Optimization and Control

Boris Houska, Yuning Jiang, Janick Frasch,

Rien Quirynen, Dimitris Kouzoupis, Moritz Diehl

ShanghaiTech University, University of Magdeburg, University of Freiburg

Motivation: sensor network localization



Decoupled case:

- each sensor takes measurement η_i of its position χ_i and solves

$$\forall i \in \{1, \ldots, 7\}, \qquad \min_{\chi_i} \|\chi_i - \eta_i\|_2^2 \ .$$

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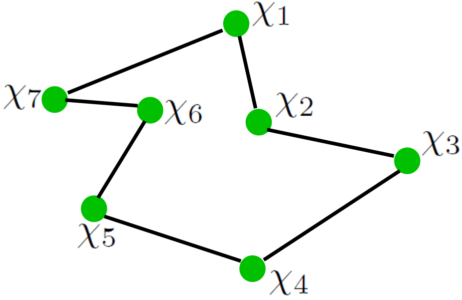

Coupled case:

- sensors additionally measure the distance to their neighbors

$$\min_{\chi} \sum_{i=1}^{7} \left\{ \|\chi_i - \eta_i\|_2^2 + \left(\|\chi_i - \chi_{i+1}\|_2 - \bar{\eta}_i\right)^2 \right\} \qquad \text{with } \chi_8 = \chi_1$$

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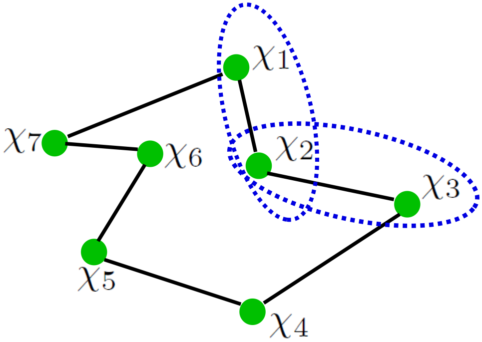

Equivalent formulation:

- set $x_1 = (\chi_1, \zeta_1)$ with $\zeta_1 = \chi_2$,
- set $x_2 = (\chi_2, \zeta_2)$ with $\zeta_2 = \chi_3$,
- and so on

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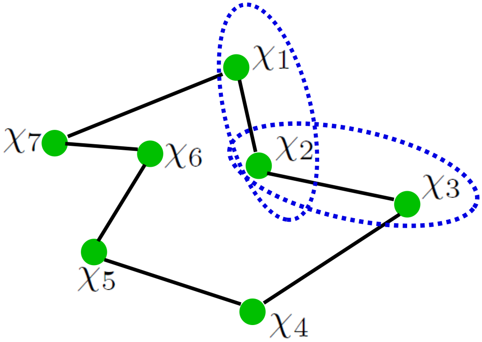

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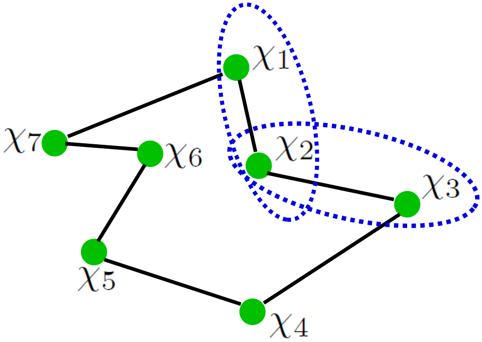

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Equivalent formulation (cont.):

- new variables $x_i = (\chi_i, \zeta_i)$
- separable non-convex objectives

$$f_i(x_i) = \frac{1}{2}\|\chi_i - \eta_i\|_2^2 + \frac{1}{2}\|\zeta_i - \eta_{i+1}\|_2^2 + \frac{1}{2}\left(\|\chi_i - \zeta_i\|_2 - \bar{\eta}_i\right)^2$$

- affine coupling, $\zeta_i = \chi_{i+1}$, can be written as

$$\sum_{i=1}^{7} A_i x_i = 0 \; .$$

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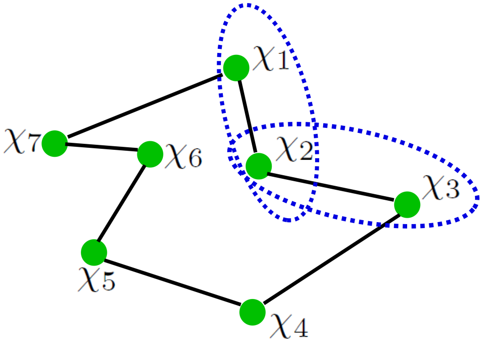

Optimization problem:

$$\min_x \sum_{i=1}^{7} f_i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^{7} A_i x_i = 0 \ .$$

Aim of distributed optimization algorithms

- Find local minimizers of

$$\min_{x} \sum_{i=1}^{N} f_i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^{N} A_i x_i = b$$

- Functions $f_i : \mathbb{R}^n \to \mathbb{R}$ potentially non-convex.
- Matrices $A_i \in \mathbb{R}^{m \times n}$ and vectors $b \in \mathbb{R}^m$ given.
- Problem: N is large.

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Overview

- **Theory**
  - Distributed optimization algorithms
  - ALADIN
- **Applications**
  - Sensor network localization
  - MPC with long horizons

Distributed optimization problem

- Find local minimizers of

$$\min_x \sum_{i=1}^{N} f_i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^{N} A_i x_i = b \; .$$

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Dual decomposition

- Main idea: solve dual problem

$$\max_{\lambda}\; d(\lambda) \quad \text{with} \quad d(\lambda) = \min_{x} \sum_{i=1}^{N} \{f_i(x_i) + \lambda^\mathsf{T} A_i x_i\} - \lambda^\mathsf{T} b$$

  Evaluation of d can be parallelized.

- Applicable if f_is are (strictly) convex
- For non-convex f_i: duality gap possible

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ADMM (consensus variant)

Alternating Direction Method of Multipliers

Input: Initial guesses $x_i \in \mathbb{R}^n$ and $\lambda_i \in \mathbb{R}^m$; $\rho > 0$, $\epsilon > 0$.

Repeat:

1. Solve decoupled NLPs

$$\min_{y_i} \; f_i(y_i) + \lambda_i^\top A_i y_i + \frac{\rho}{2} \left\| A_i(y_i - x_i) \right\|_2^2 \; .$$

2. Implement dual gradient steps $\lambda_i^+ = \lambda_i + \rho A_i(y_i - x_i)$.

3. Solve coupled QP

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4. Update the iterates $x \leftarrow x^+$ and $\lambda \leftarrow \lambda^+$.

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Limitations of ADMM

1) Convergence rate of ADMM is very scaling dependent.

2) ADMM may be divergent, if f_is are nonconvex. Example:

$$\min_x \; x_1 \cdot x_2 \qquad \text{s.t.} \qquad x_1 - x_2 = 0 \; .$$

- unique and regular minimizer at $x_1^* = x_2^* = \lambda^* = 0$.
- For $\rho = \frac{3}{4}$ all sub-problems are strictly convex.
- ADMM is divergent; $\lambda^+ = -2\lambda$.

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ALADIN (full step variant)

Augmented Lagrangian based Alternating Direction Inexact Newton Method

Input:

- Initial guesses $x_i \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$, $\rho > 0$, $\epsilon > 0$.

Repeat:

1. Solve decoupled NLPs

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2. Compute $g_i = \nabla f_i(y_i)$, choose $H_i \approx \nabla^2 f_i(y_i)$, and solve coupled QP

$$\min_{\Delta y} \; \sum_{i=1}^{N} \left\{ \frac{1}{2} \Delta y_i^\mathsf{T} H_i \Delta y_i + g_i^\mathsf{T} \Delta y_i \right\} \quad \text{s.t.} \quad \sum_{i=1}^{N} A_i (y_i + \Delta y_i) = b \;\mid\; \lambda^+$$

3. Set $x \leftarrow y + \Delta y$ and $\lambda \leftarrow \lambda^+$.

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Special cases

- For $\rho \to \infty$:

$$\text{ALADIN} \;\equiv\; \text{SQP}$$

- For $0 < \rho < \infty$:

If $H_i = \rho A_i^\top A_i$, $\Sigma_i = A_i^\top A_i$, then

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Local convergence

Assumptions

- f_is are twice continuously differentiable
- minimizer (x^*, λ^*) regular KKT point (LICQ + SOSC satisfied)
- ρ satisfies $\nabla^2 f_i(y_i) + \rho\Sigma_i \succ 0$

Theorem Full-step variant of ALADIN converges locally

1. with quadratic convergence rate, if $H_i = \nabla^2 f_i(y_i) + \mathbf{O}(\|y_i - x^*\|)$
2. with linear converges rate, if $\|H_i - \nabla^2 f_i(y_i)\|$ is sufficiently small

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Globalization

Definition (L1-penalty function)

- We say that x^+ is a descent step if $\Phi(x^+) < \Phi(x)$ for

$$\Phi(x) \; = \; \sum_{i=1}^{N} f_i(x_i) + \overline{\lambda} \left\| \sum_{i=1}^{N} A_i x_i - b \right\|_1 \; ,$$

  $\overline{\lambda}$ sufficiently large.

Globalization

Rough sketch:

- As long as $\nabla^2 f_i(x_i) + \rho\Sigma_i \succ 0$ the proximal objectives

$$\tilde{f}_i(y_i) = f_i(y_i) + \frac{\rho}{2} \|y_i - x_i\|_{\Sigma_i}^2$$

  are strictly convex in a neighborhood of the current primal iterates x_i.

- If we don't update x, y can be enforced to converge to the solution z of the convex auxiliary problem

$$\min_z \sum_{i=1}^{N} \tilde{f}_i(z_i) \quad \text{s.t.} \quad \sum_{i=1}^{N} A_i z_i = b\,.$$

- Strategy: if x^+ is not a descent direction, we skip the primal update until y is a descent and set $x^+ = y$ (similar to proximal methods)
- This strategy leads to a globalization routine for ALADIN if $H_i \succ 0$.

Globalization

Rough sketch:

- As long as $\nabla^2 f_i(x_i) + \rho\Sigma_i \succ 0$ the proximal objectives

$$\tilde{f}_i(y_i) = f_i(y_i) + \frac{\rho}{2}\left\|y_i - x_i\right\|_{\Sigma_i}^2$$

  are strictly convex in a neighborhood of the current primal iterates x_i.

- If we don't update x, y can be enforced to converge to the solution z of the convex auxiliary problem

$$\min_z \sum_{i=1}^{N} \tilde{f}_i(z_i) \quad \text{s.t.} \quad \sum_{i=1}^{N} A_i z_i = b \; .$$

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Inequalities

- Distributed NLP with inequalities:

$$\min_{x} \sum_{i=1}^{N} f_i(x_i) \quad \text{s.t.} \quad \left\{ \begin{array}{l} \sum_{i=1}^{N} A_i x_i = b \\ h_i(x_i) \leq 0 \ . \end{array} \right.$$

- Functions $f_i : \mathbb{R}^n \to \mathbb{R}$ and $h_i : \mathbb{R}^n \to \mathbb{R}^{n_h}$ potentially non-convex.
- Matrices $A_i \in \mathbb{R}^{m \times n}$ and vectors $b \in \mathbb{R}^m$ given.
- Problem: N is large.

ALADIN (with inequalities)

Augmented Lagrangian based Alternating Direction Inexact Newton Method

Input:

- Initial guesses $x_i \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$, $\rho > 0$, $\epsilon > 0$.

Repeat:

1. Solve decoupled NLPs

$$\min_{y_i} \; f_i(y_i) + \lambda^\mathsf{T} A_i y_i + \frac{\rho}{2} \| y_i - x_i \|^2_{\Sigma_i} \quad \text{s.t.} \quad h_i(y_i) \leq 0 \;\mid\; \kappa_i \,.$$

2. Set $g_i = \nabla f_i(y_i) + \nabla h_i(y_i)\kappa_i$, $H_i \approx \nabla^2 \left(f_i(y_i) + \kappa_i^\mathsf{T} h_i(y_i) \right)$, solve

$$\min_{\Delta y} \; \sum_{i=1}^{N} \left\{ \frac{1}{2} \Delta y_i^\mathsf{T} H_i \Delta y_i + g_i^\mathsf{T} \Delta y_i \right\} \quad \text{s.t.} \quad \sum_{i=1}^{N} A_i (y_i + \Delta y_i) = b \;\mid\; \lambda^+$$

3. Set $x \leftarrow y + \Delta y$ and $\lambda \leftarrow \lambda^+$.

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ALADIN (with inequalities)

Augmented Lagrangian based Alternating Direction Inexact Newton Method

Remarks:

- If approximation $C_i \approx C_i^* = \nabla h_i(y_i)$ is available, solve QP

$$\min_{\Delta y, s} \quad \sum_{i=1}^{N} \left\{ \frac{1}{2} \Delta y_i^\mathsf{T} H_i \Delta y_i + g_i^\mathsf{T} \Delta y_i \right\} + \lambda^\mathsf{T} s + \frac{\mu}{2} \|s\|_2^2$$

$$\text{s.t.} \qquad \left\{ \begin{array}{rcl} \sum_{i=1}^{N} A_i (y_i + \Delta y_i) & = & b + s \quad \Big| \; \lambda_{\mathrm{QP}} \\ C_i \Delta y_i & = & 0 \,, \quad i \in \{1, \ldots, N\} \,. \end{array} \right.$$

  with $g_i = \nabla f_i(y_i) + (C_i^* - C_i)^\mathsf{T} \kappa_i$ and $\mu > 0$ instead.

- If H_i and C_i constant, pre-compute matrix decompositions

ALADIN (with inequalities)

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$$
\begin{aligned}
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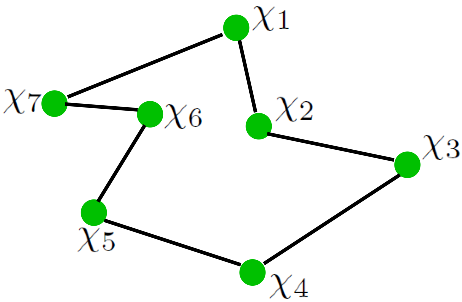
Overview

- **Theory**
  - Distributed Optimization Algorithms
  - ALADIN
- **Applications**
  - Sensor network localization
  - MPC with long horizons

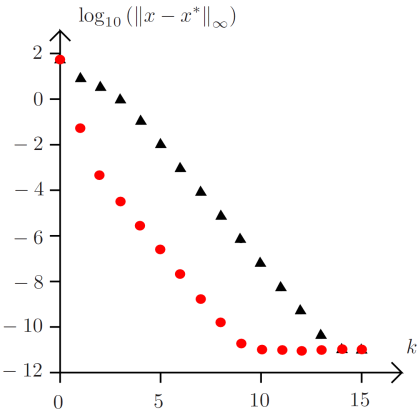
Sensor network localization



Case study:

- 25000 sensors measure positions and distances in a graph
- additional inequality constraints, $\|\chi_i - \eta_i\|_2 \leq \bar{\sigma}$, remove outliers.

ALADIN versus SQP



- 10^5 primal and $7.5 * 10^4$ dual optimization variables
- implementation in `JULIA`

Overview

- **Theory**
  - Distributed Optimization Algorithms
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- **Applications**
  - Sensor network localization
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Nonlinear MPC

Repeat:

- Wait for state measurement $\hat{x}$.
- Solve

$$\min_{x,u} \quad \sum_{i=0}^{m-1} l(x_i, u_i) + M(x_m)$$

$$\text{s.t.} \quad \begin{cases} x_{i+1} = f(x_i, u_i) \\ x_0 = \hat{x} \\ (x_i, u_i) \in \mathbb{X} \times \mathbb{U}\ , \end{cases}$$

- Send u_0 to the process.

ACADO Toolkit

Mathematical Formulation

$$
\begin{aligned}
\min_{x,u} \quad & \int_0^T x^2 + u^2 \mathrm{d}t \\
& \dot{x} = f(x,u) \\
\text{s.t.} \quad & x(0) = x_0 \\
& -1 \leq u \leq 1\ .
\end{aligned}
$$

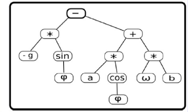
ACADO Syntax

```
DifferentialState      x;
Control                u;

DifferentialEquation f;
f << dot(x) == u + ...;

ocp.minLagrangeTerm( x*x+u*u );
ocp.subjectTo( f );
ocp.subjectTo( -1 <= u <= 1 );
```

Symbolic Structure Detection



Algorithm

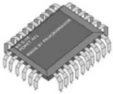
- Multiple Shooting
- Real-Time Gauss Newton
- Online Active Set Strategy

Optimized C-Code

```
r[1] = a[15]*c[17] + a[16]*c[19] + ... ;
r[2] = sin(a[1]*a[2]) + a[4] + ... ;
r[3] = cos(r[1])/exp(c[4])+ r[1] +... ;
```

Customized Solver Implemented on Chip/FPGA:

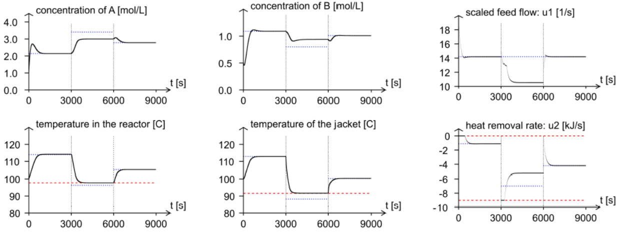
Measurement x_0



Optimal Decision u*

MPC Benchmark

Nonlinear Model, 4 States, 2 Controls



Run-time of ACADO code generation (not distributed)

	CPU time	Percentage
Integration & sensitivities	$117\ \mu$s	$65\,\%$
QP (Condensing + qpOASES)	$59\ \mu$s	$33\,\%$
Complete real-time iteration	$181\ \mu$s	$100\,\%$

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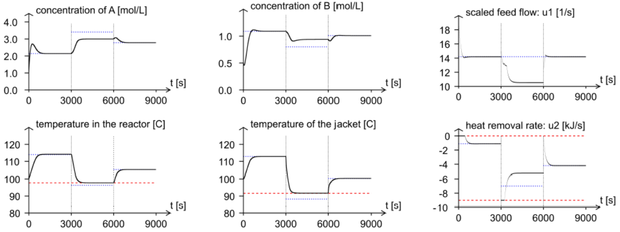

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MPC with ALADIN

ALADIN Step 1: solve decoupled NLPs

- choose a short horizon $n = \frac{m}{N}$ and solve

$$\min_{y,v} \quad \Psi_j(y_0^j) + \sum_{i=0}^{n-1} l(y_i^j, v_i^j) + \Phi_{j+1}(y_n^j)$$

$$\text{s.t.} \quad \left\{ \begin{array}{l} y_{i+1}^j = f(y_i^j, v_i^j) \ , i = 0, ..., n-1 \\ y_i^j \in \mathbb{X} \ , \ v_i^j \in \mathbb{U} \ . \end{array} \right.$$

- Arrival and end costs depend on ALADIN iterates,

$$\begin{array}{rclcrcl} \Psi_0(y) & = & I(\hat{x}, y) & & \Phi_N(y) & = & M(y) \\ \Psi_j(y) & = & -\lambda_j^\mathsf{T} y + \frac{\rho}{2} \|y - z_j\|_P^2 & & \Psi_j(y) & = & \lambda_j^\mathsf{T} y + \frac{\rho}{2} \|y - z_j\|_P^2 \end{array}$$

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MPC with ALADIN

ALADIN Step 2: solve coupled QP

- As we have no constraints, QP $=$ LQR problem
- solve all matrix-valued Ricatti equations offline
- solve online QP online by backward-forward sweep

Code export

- export all online operations as optimized C-code
- NLP solver: explicit MPC (rough heuristic)
- one ALADIN iteration per sampling time, skip globalization

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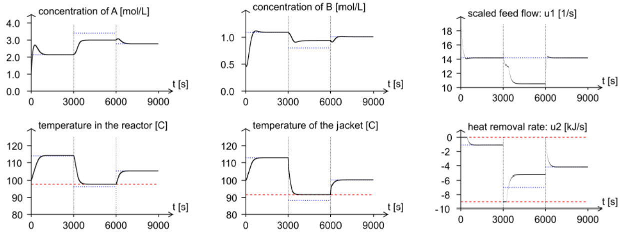
ALADIN + code generation (results by Yuning Jiang)

Same nonlinear model as before, $n = 2$



Run-time of real-time ALADIN, 10 processors

	CPU time	Percentage
Parallel explicit MPC	6 μs	54 %
QP sweeps	3 μs	27 %
communication overhead	2 μs	18 %

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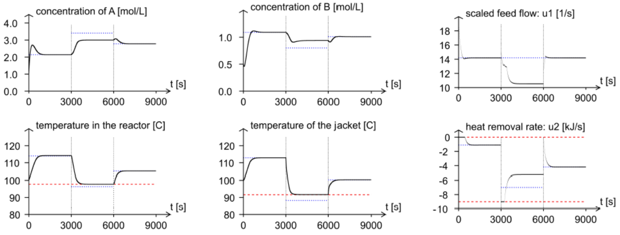

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- can solve nonconvex distributed optimization problems,

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- local convergence analysis similar to SQP; globalization possible

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