LMI-based Robust MPC Design

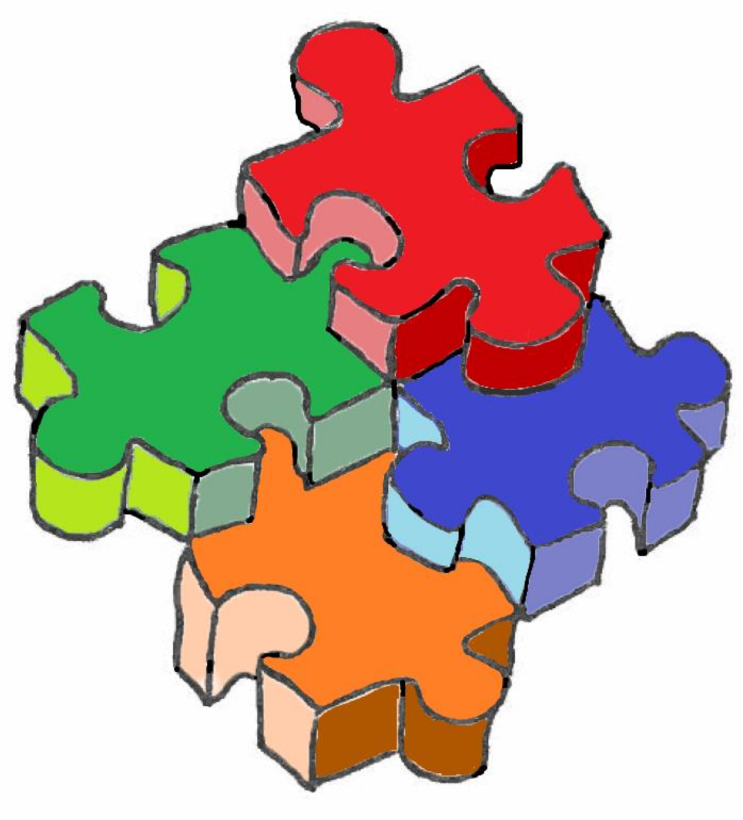
Alternative Robust MPC Design

Juraj Oravec – Monika Bakošová

STU

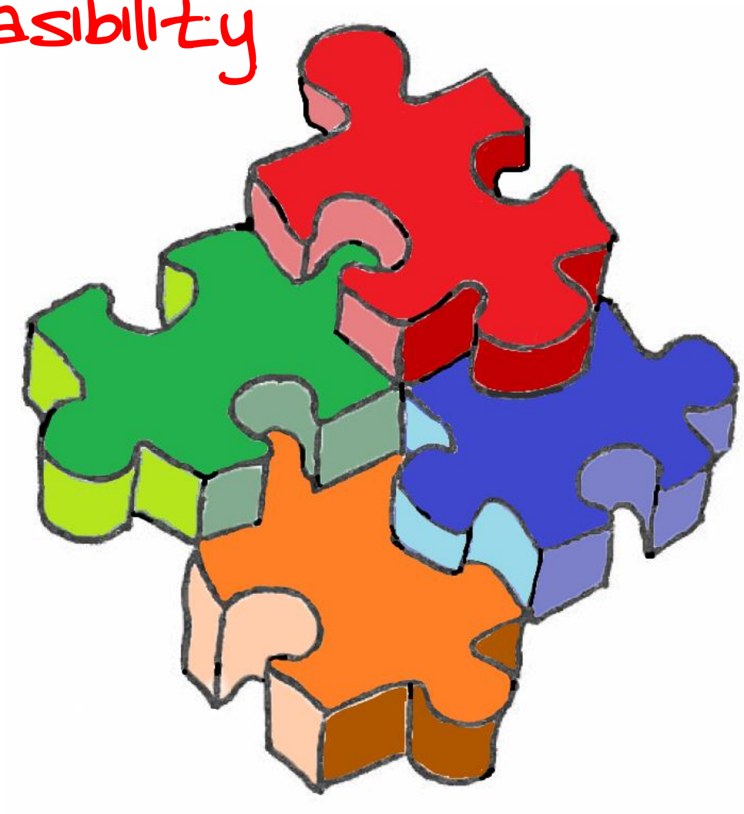
Slovak University of Technology in Bratislava

Alternative Approaches



Alternative Approaches

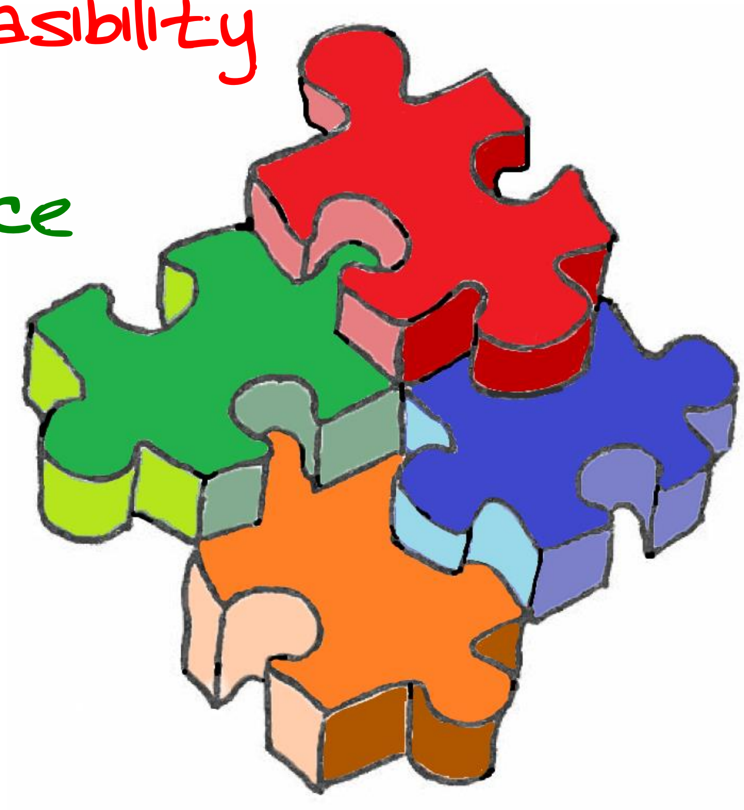
feasibility



Alternative Approaches

feasibility

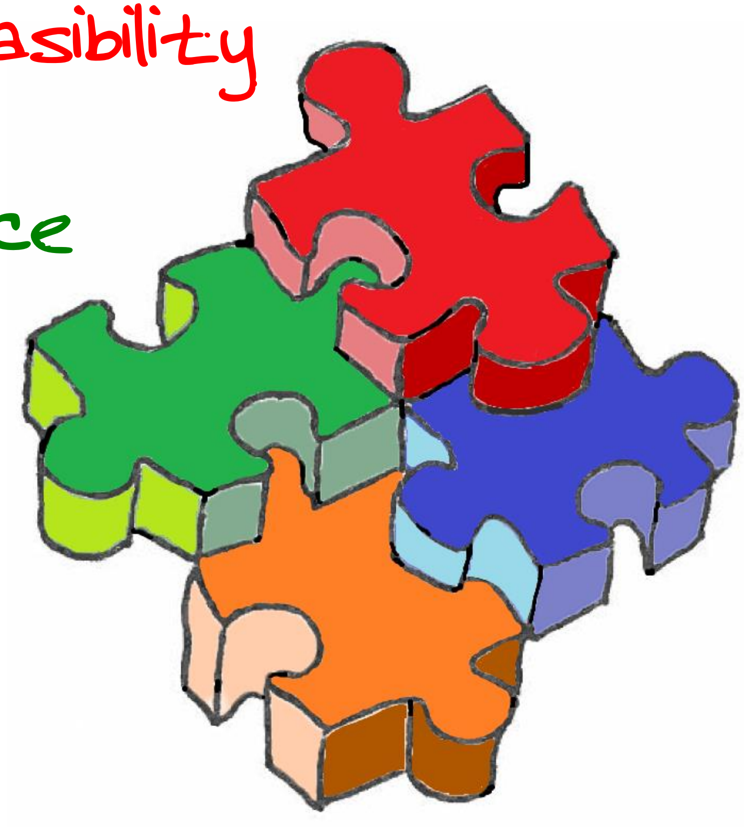
performance



Alternative Approaches

feasibility

performance

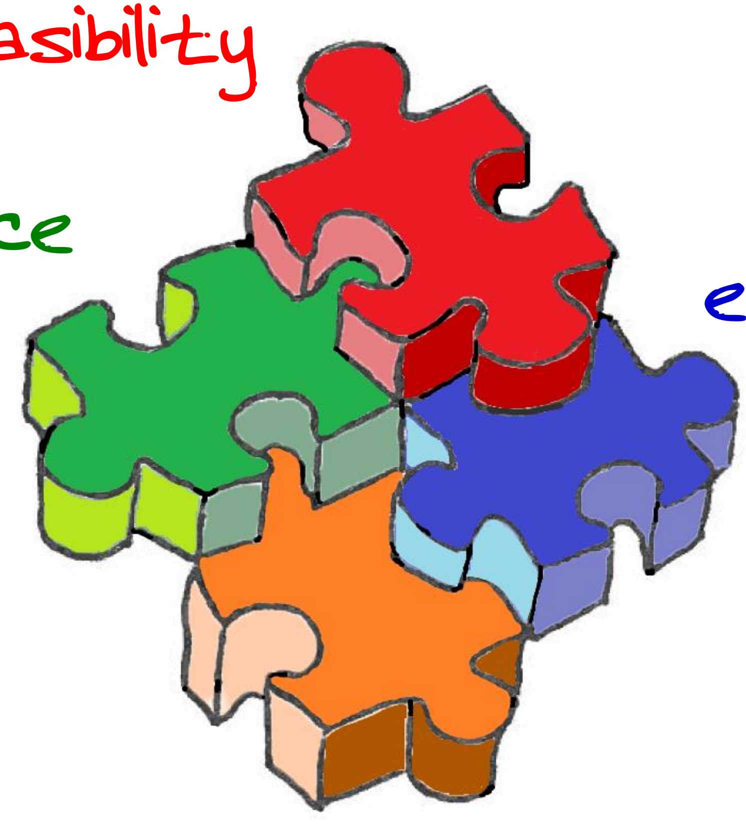


conservativeness

Alternative Approaches

feasibility

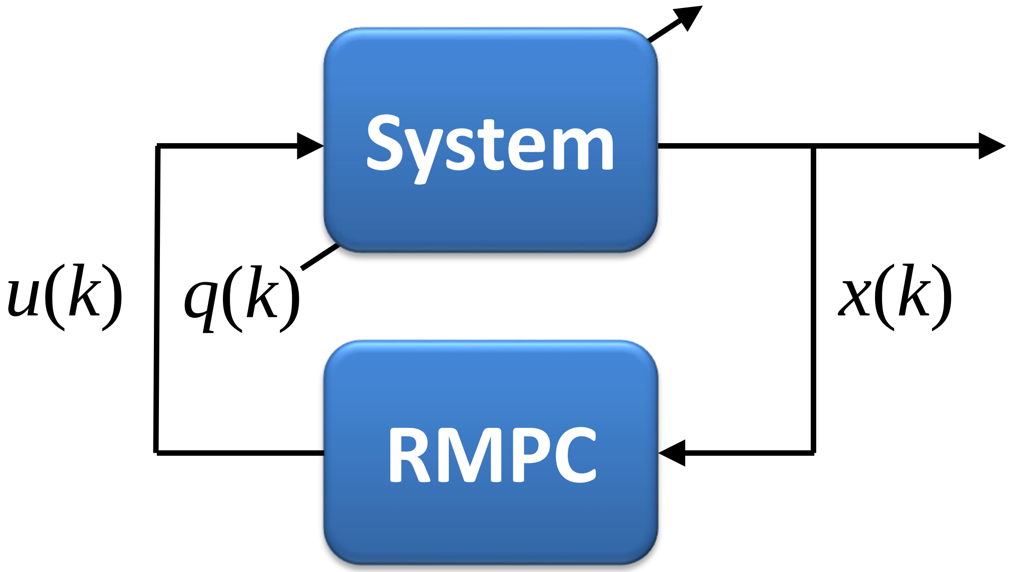
performance



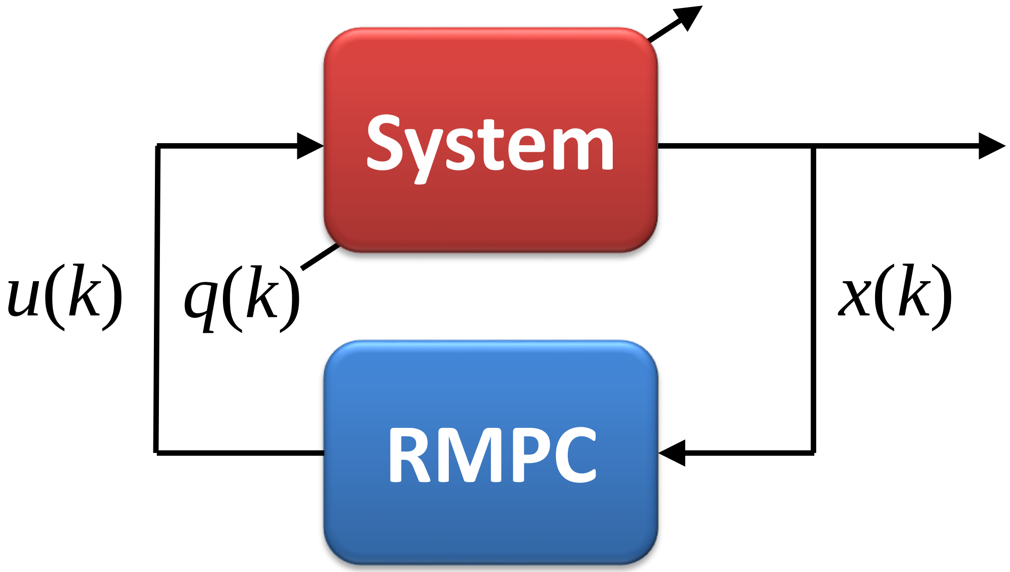
efficiency

conservativeness

Problems to solve

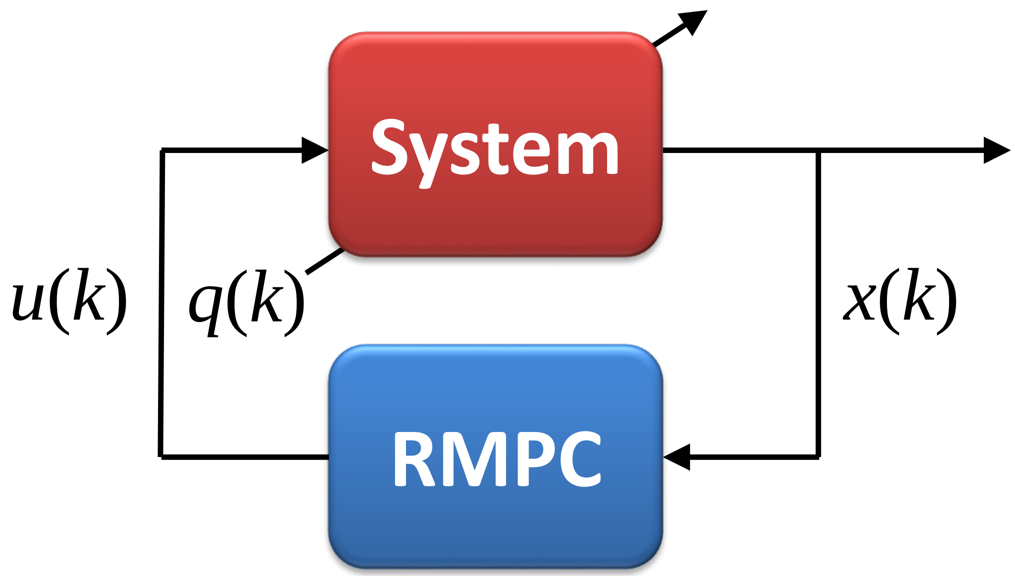


Problems to solve



$$x(k+1) = A\,x(k) + B\,u(k), \qquad x(0) = x_0,$$
$$y(k) = C\,x(k),$$

Problems to solve



$$x(k+1) = A\,x(k) + B\,u(k), \qquad x(0) = x_0,$$
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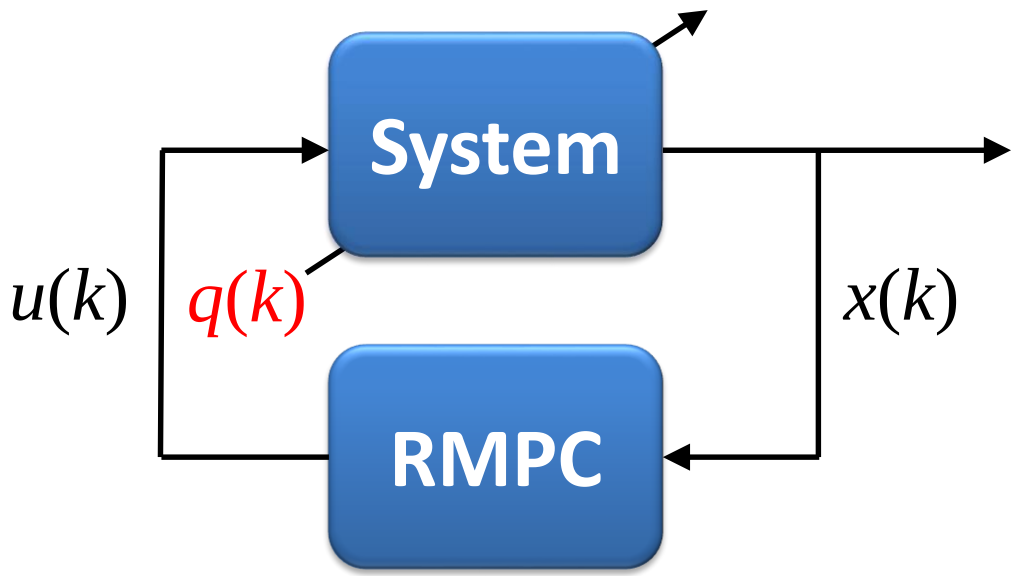

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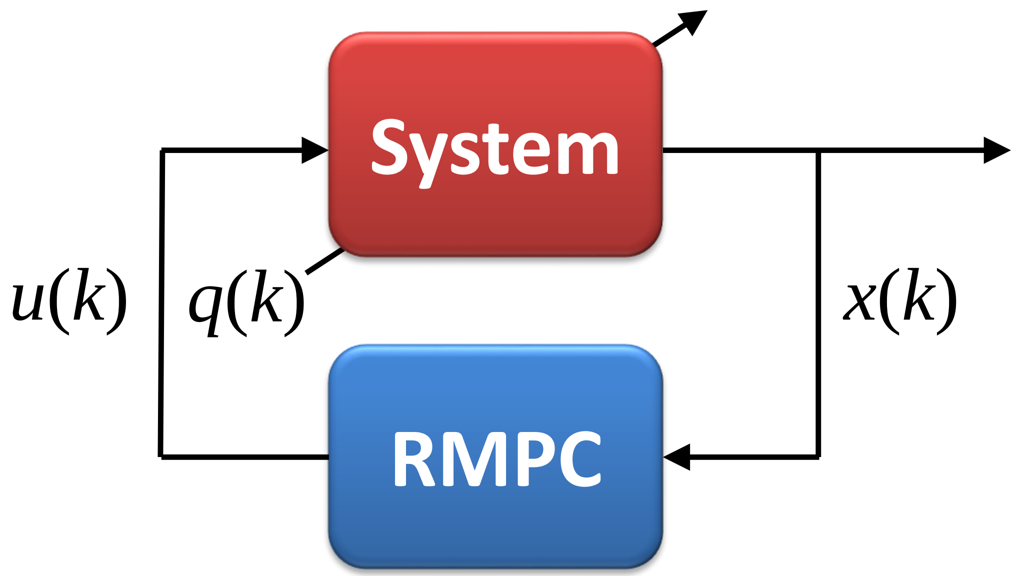
$$q \in \mathbb{Q}.$$

Problems to solve

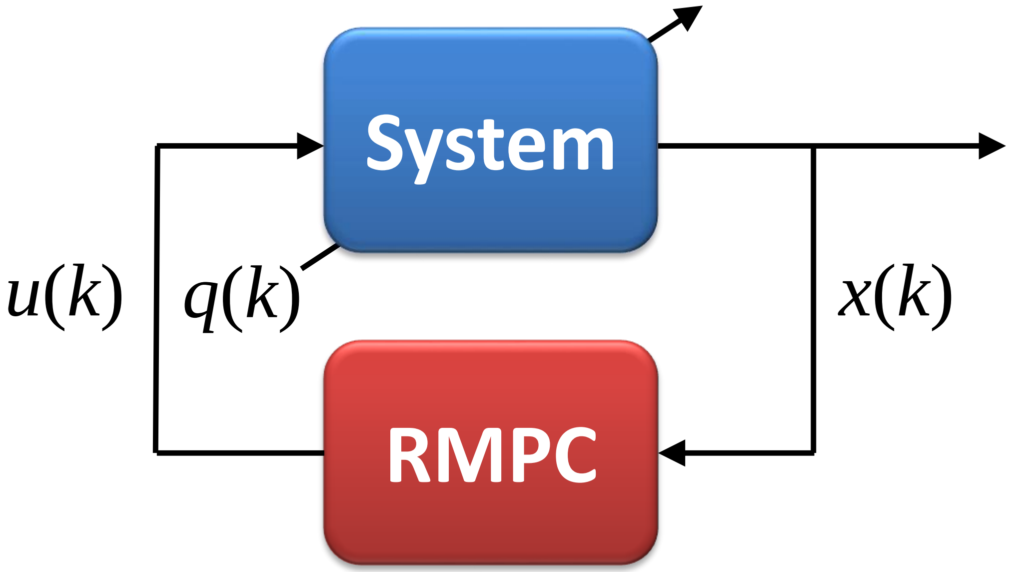


$$x(k+1) = A\,x(k) + B\,u(k), \qquad x(0) = x_0,$$

$$y(k) = C\,x(k), \qquad u \in \mathbb{U},\ y \in \mathbb{Y},$$

$$[A, B] \in \mathrm{convhull}\{[A_v, B_v]\}, \qquad q \in \mathbb{Q}.$$

Analysis

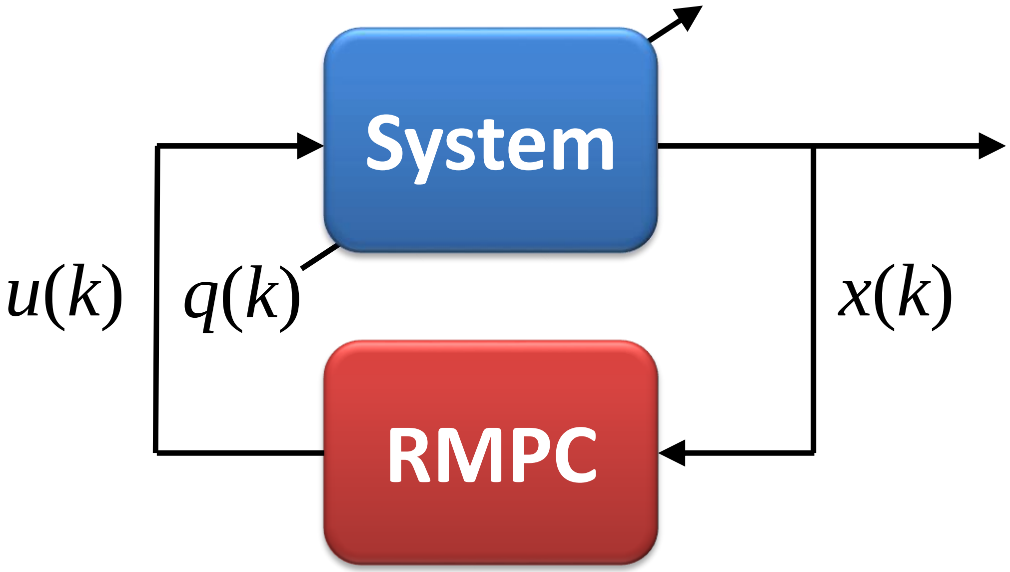


SDP:

$$\min \; c^\top x$$

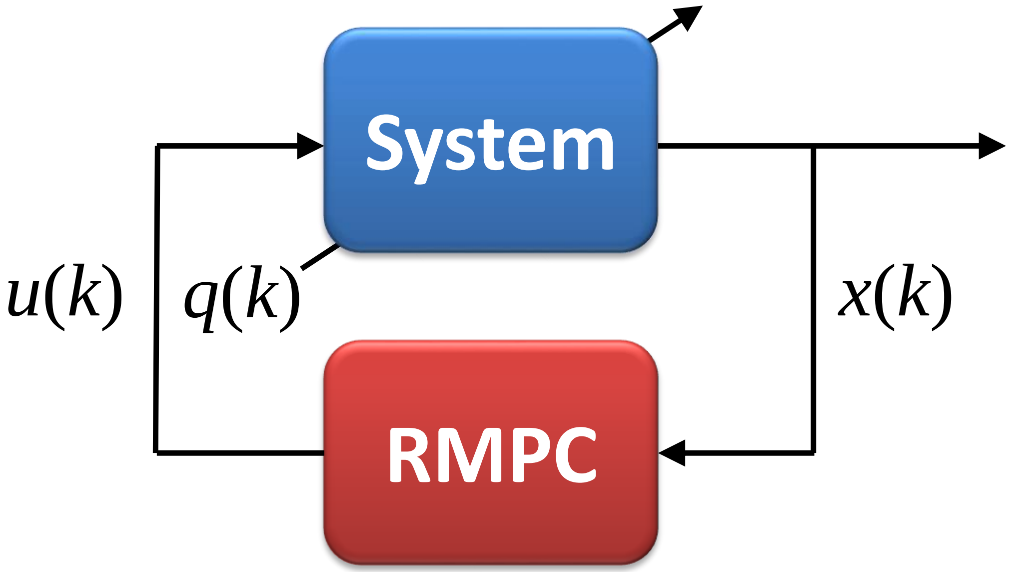
$$\text{s.t.: } \; x \preceq_{\mathcal{K}} 0 \Rightarrow M(x) \prec 0$$

Synthesis



Control law: $u(k) = F(x(k))\,x(k)$

Synthesis



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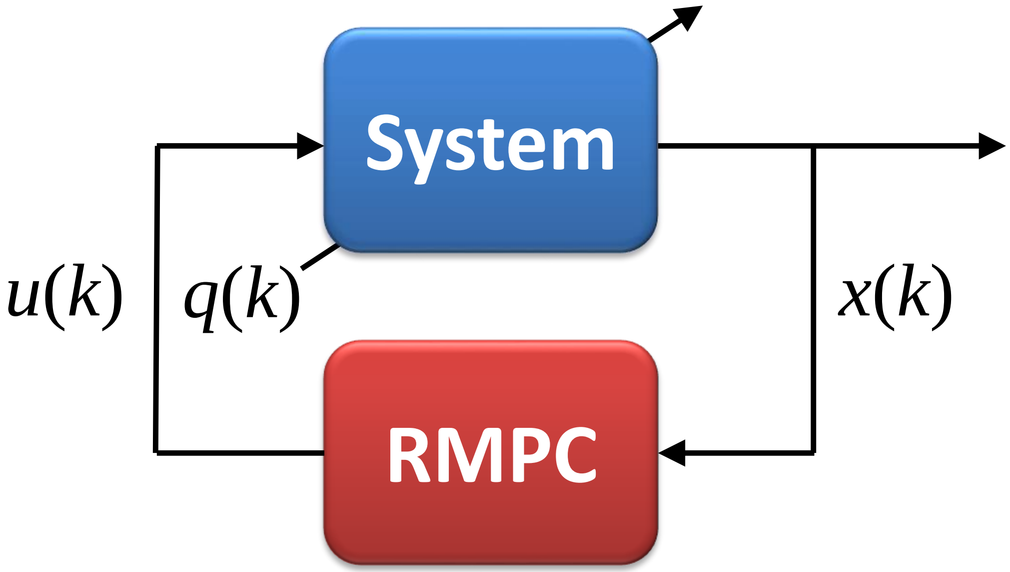
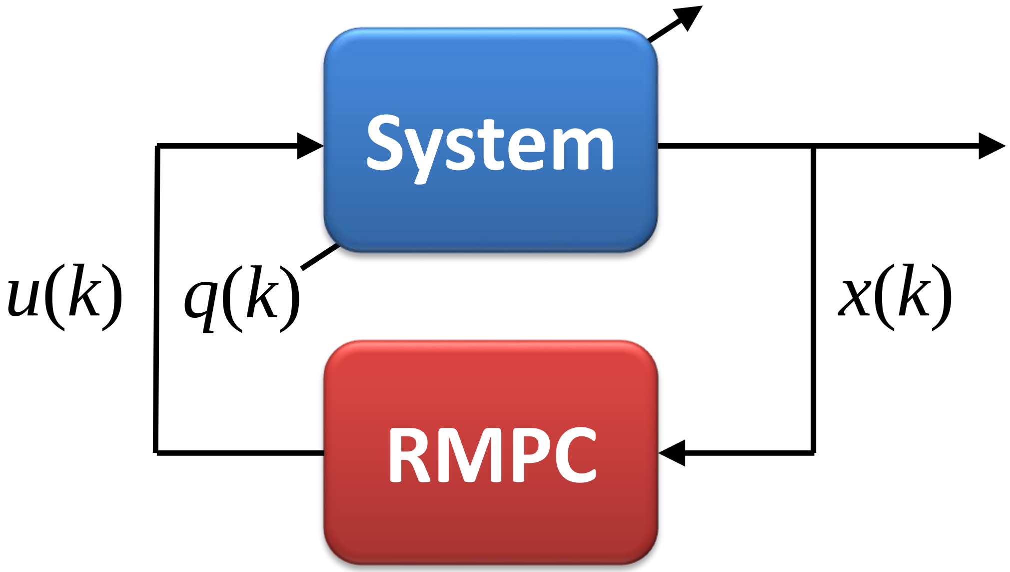


Control law: $u(k) = F(x(k))\, x(k)$

Quality criterion:

$$J(k) = \sum_{i=0}^{\infty} \left(x(k+i)^\top Q\, x(k+i) + u(k+i)^\top R\, u(k+i) \right)$$

Synthesis



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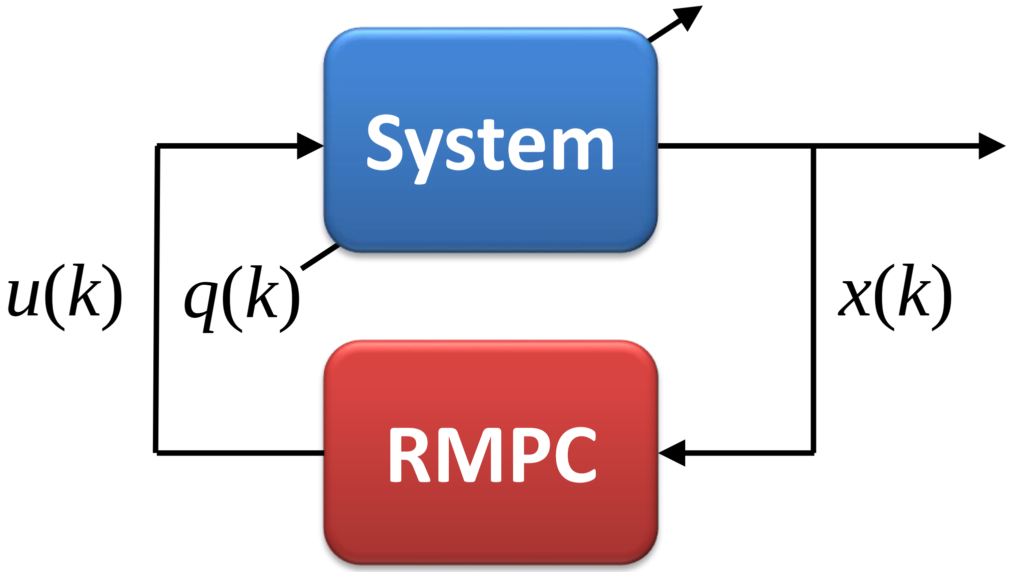
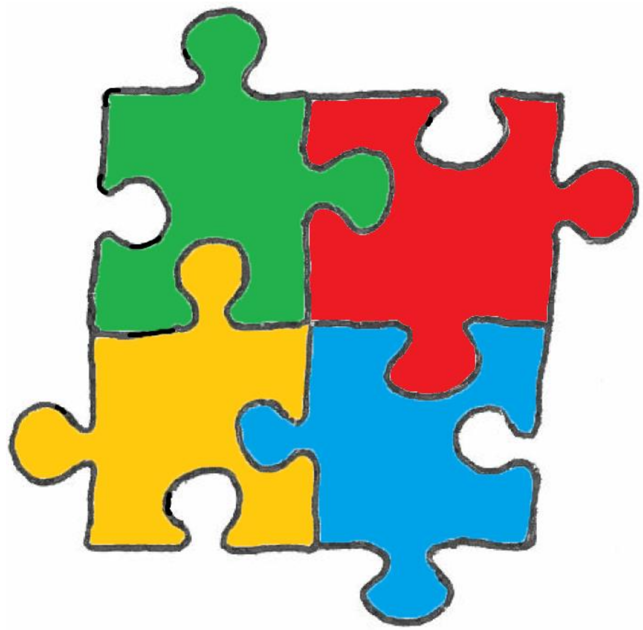
Synthesis



Control law: $u(k) = F(x(k))\,x(k)$

Quality criterion:

$$J(k) = \sum_{i=0}^{\infty} \ell(x(k+i), u(k+i))$$



Lyapunov Function

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$$V(x(k)) = x(k)^\top P x(k),\ 0 \prec P = P^\top \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$$

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Lyapunov Function

Crucial Scenario

Lyapunov Function

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$$V(x(k+1)) - V(x(k)) \leq -\ell(u(k), x(k))$$

Lyapunov Function

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Lyapunov Function

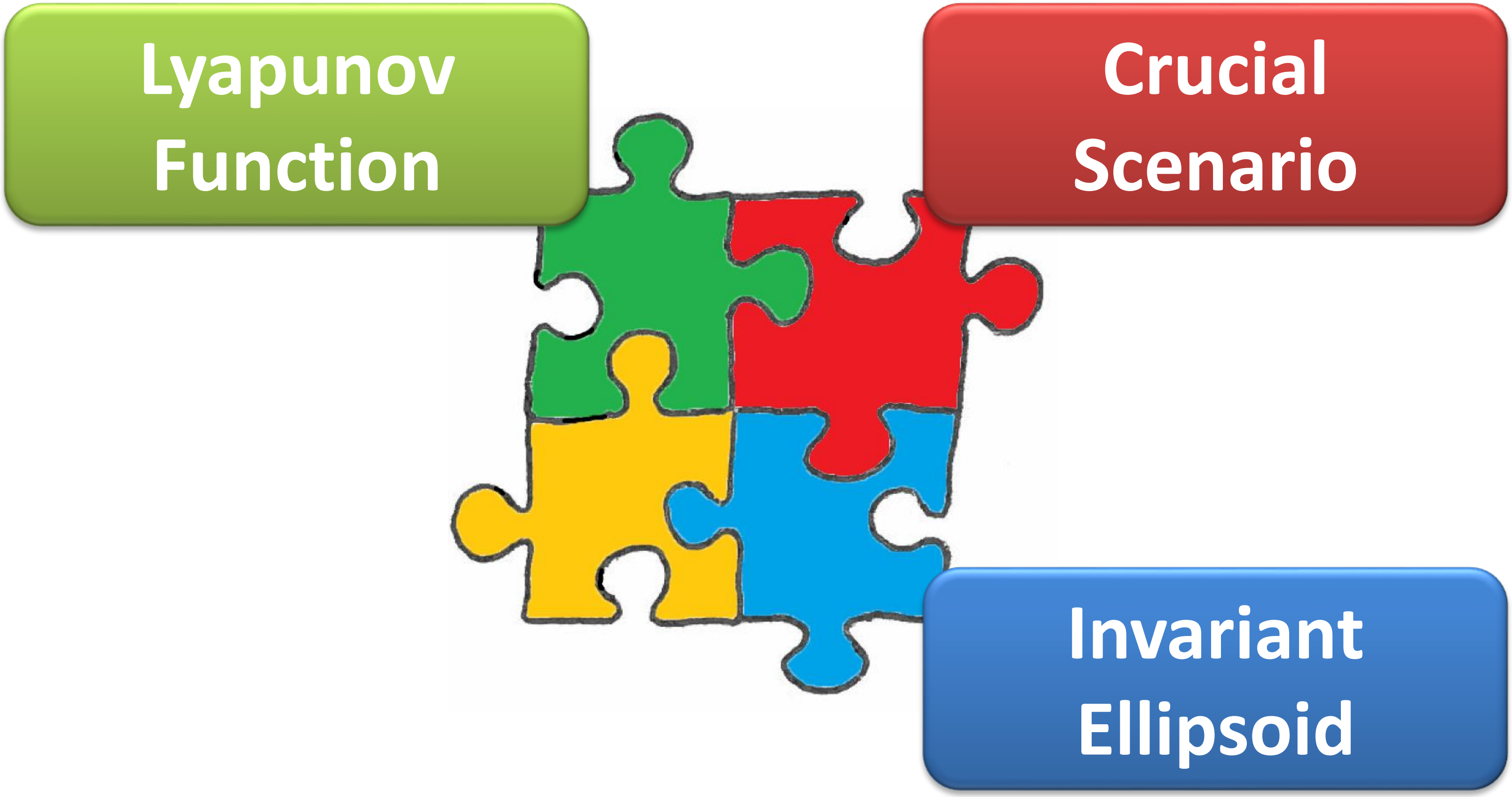
Crucial Scenario

$$V(x(k+1)) - V(x(k)) \leq -\ell(u(k), x_{\text{worst}}(k))$$

Lyapunov Function

Crucial Scenario

$$V(x(k+1)) - V(x(k)) \leq -\ell(u(k), x_{\mathrm{nom}}(k))$$


Lyapunov Function
Crucial Scenario
Invariant Ellipsoid

Lyapunov Function

Crucial Scenario

Invariant Ellipsoid

$$\varepsilon = \{x \mid x(k)^\top P x(k) \leq \gamma\}$$

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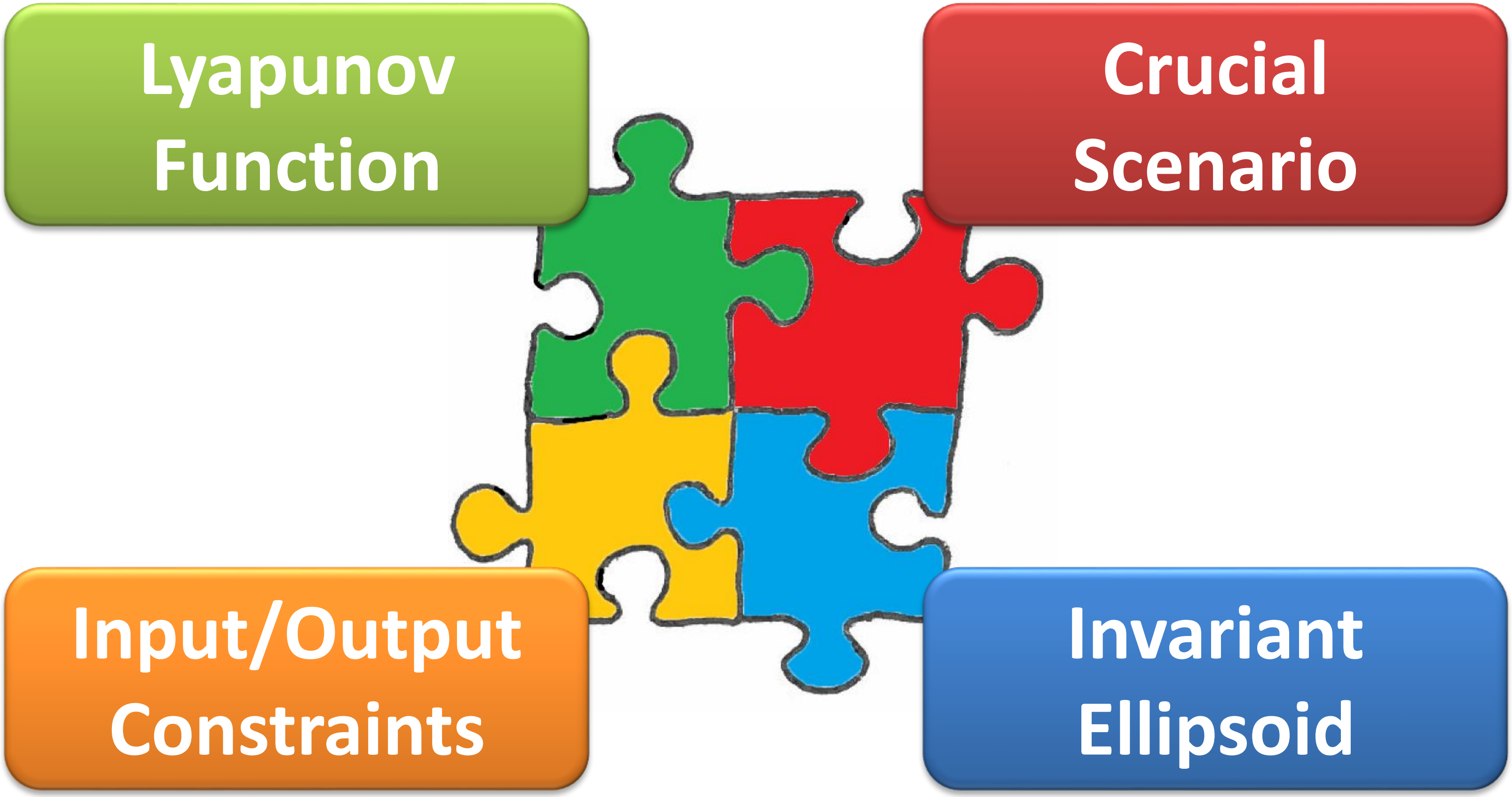
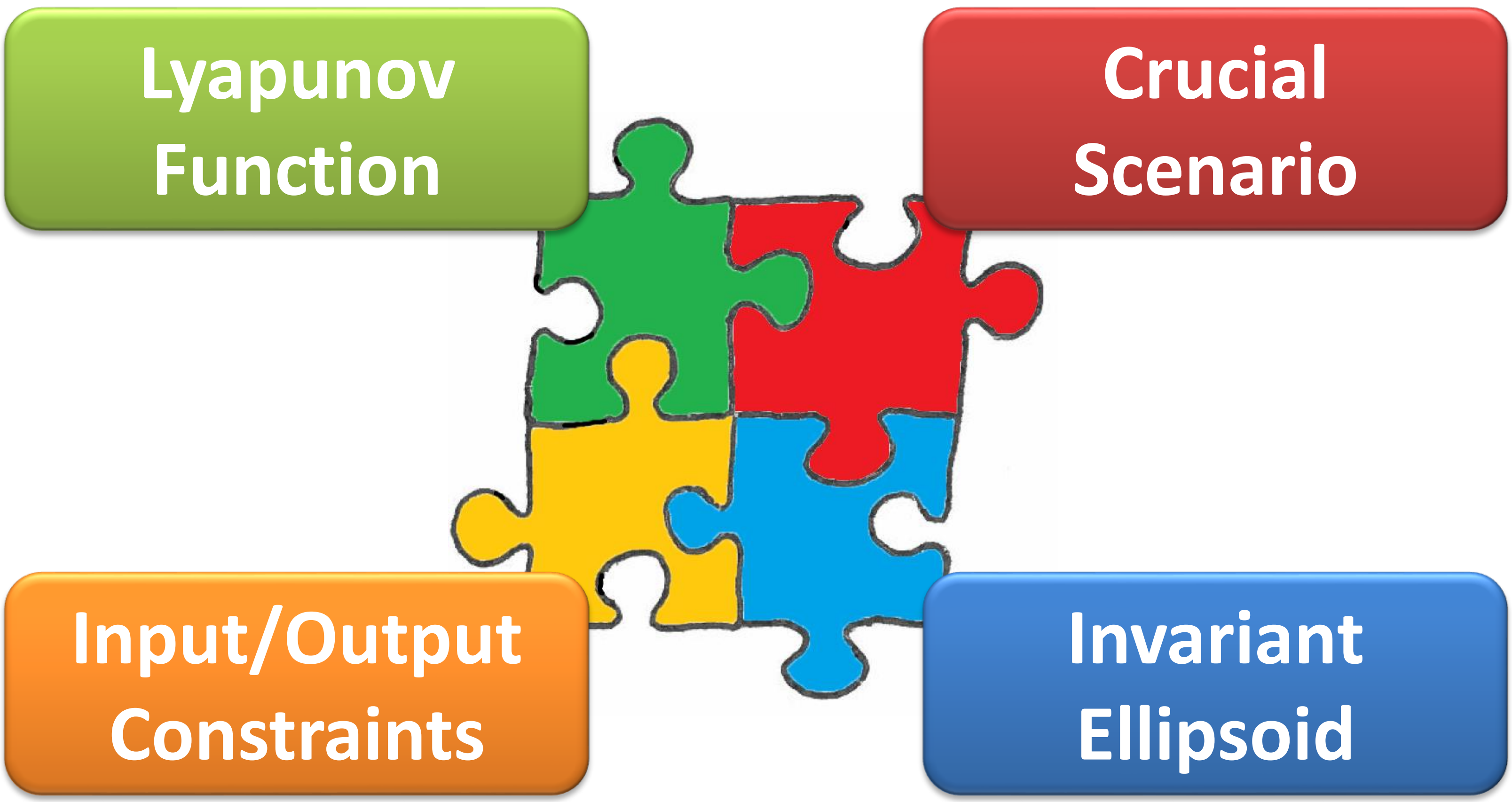
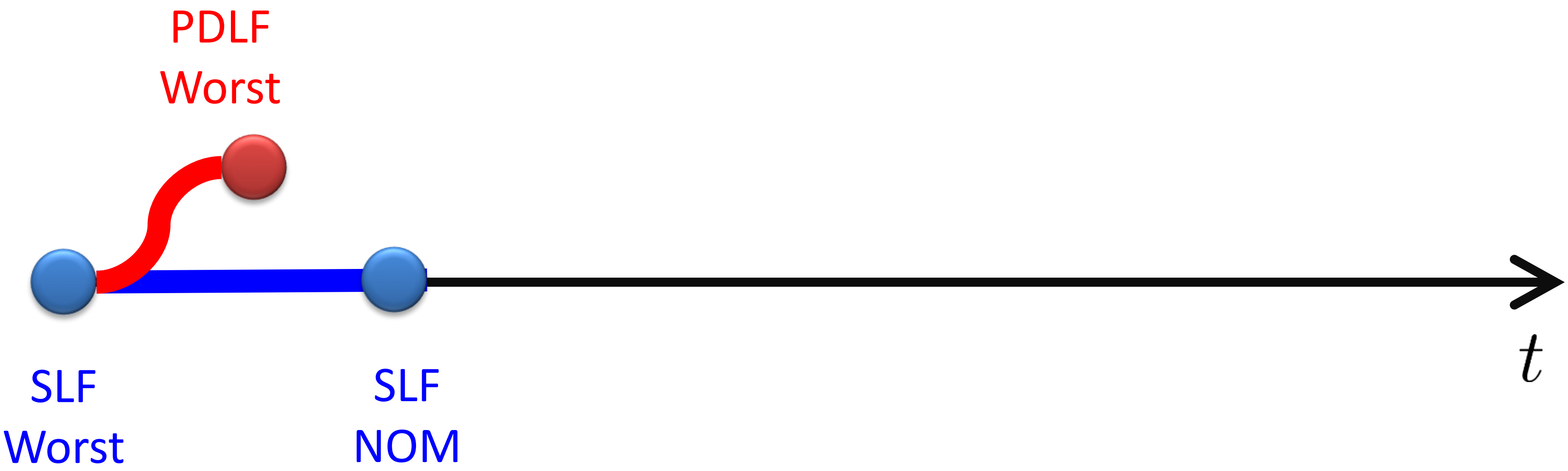

Lyapunov Function
Crucial Scenario
Input/Output Constraints
Invariant Ellipsoid



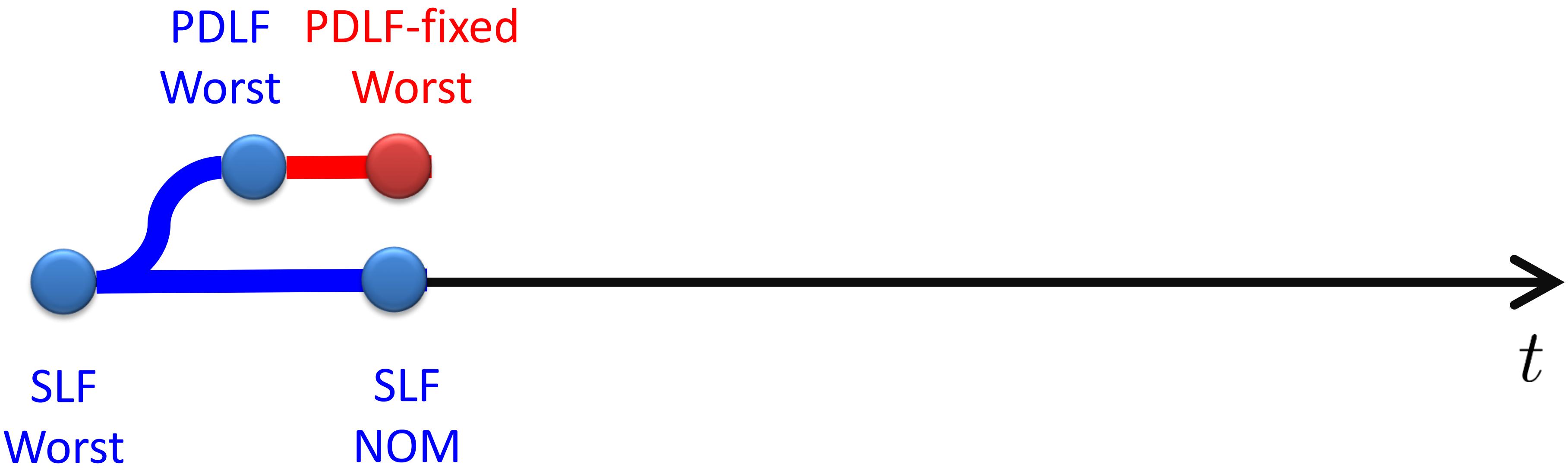
$$\|u(k)\|^2 \leq \|u_{\max}\|^2, \qquad \|y(k)\|^2 \leq \|y_{\max}\|^2$$

Kothare et al., Automatica, 1996.

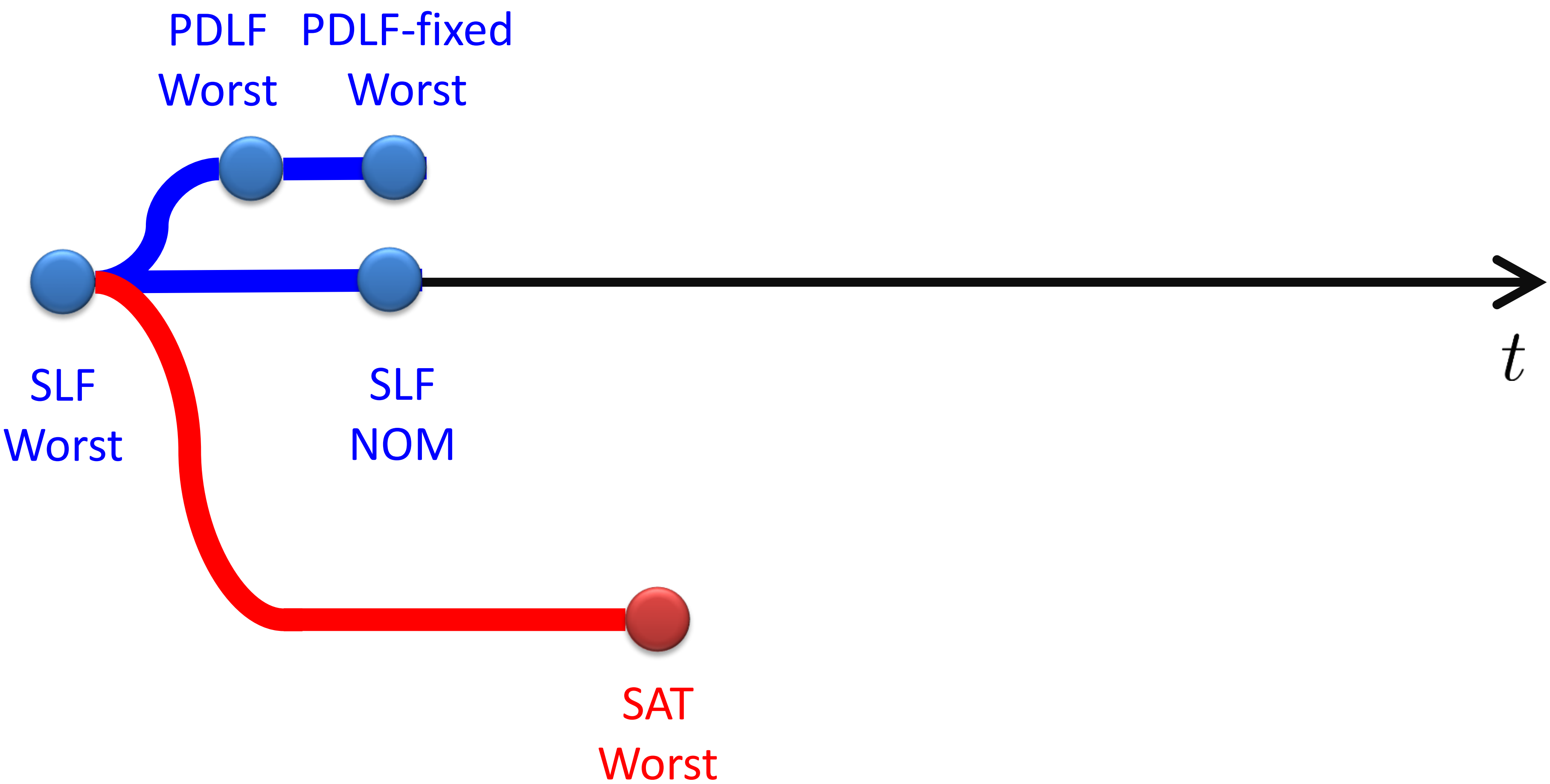
Wan and Kothare, Automatica, 2003.



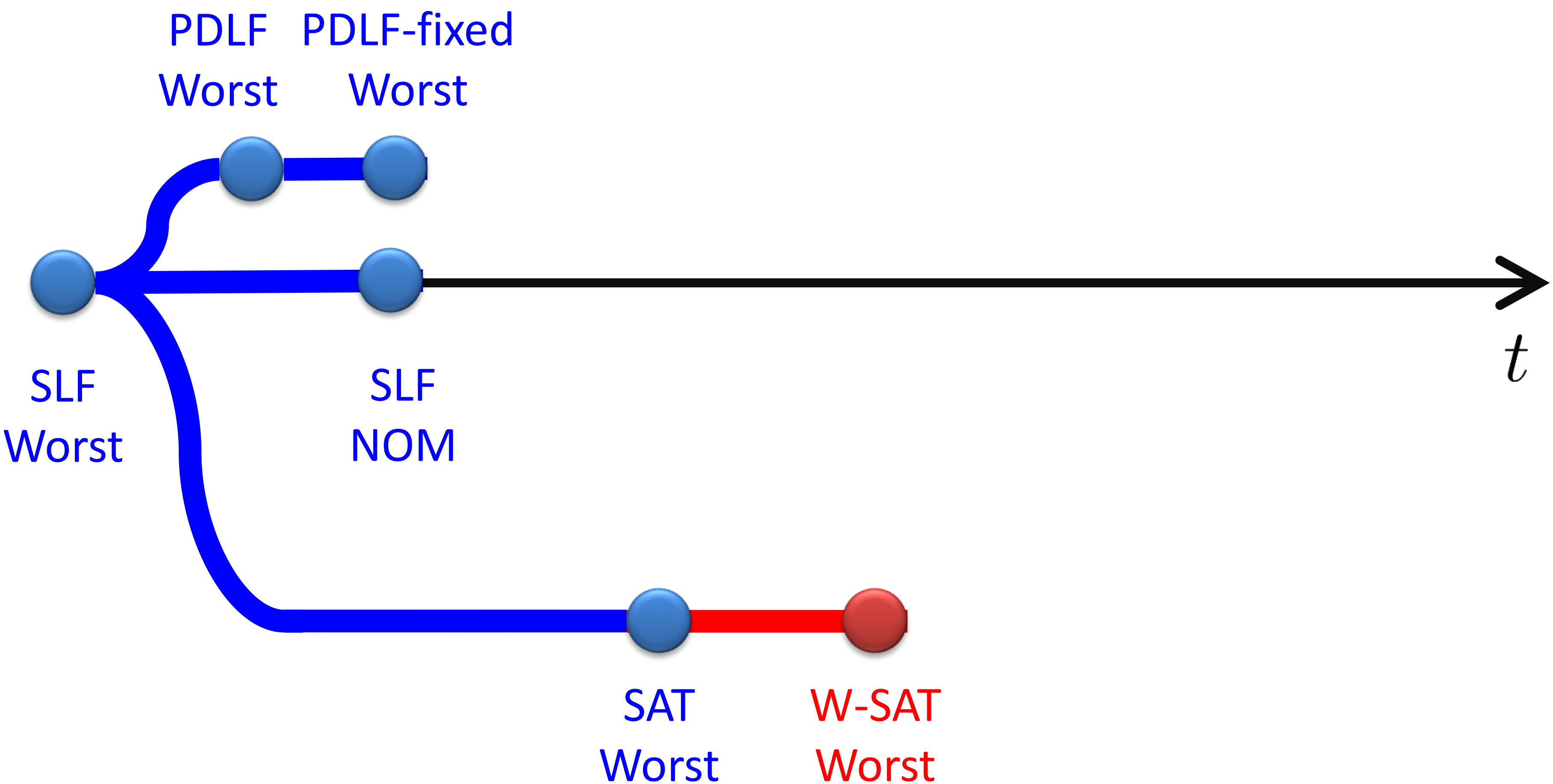
Cuzzola et al., Automatica, 2002.



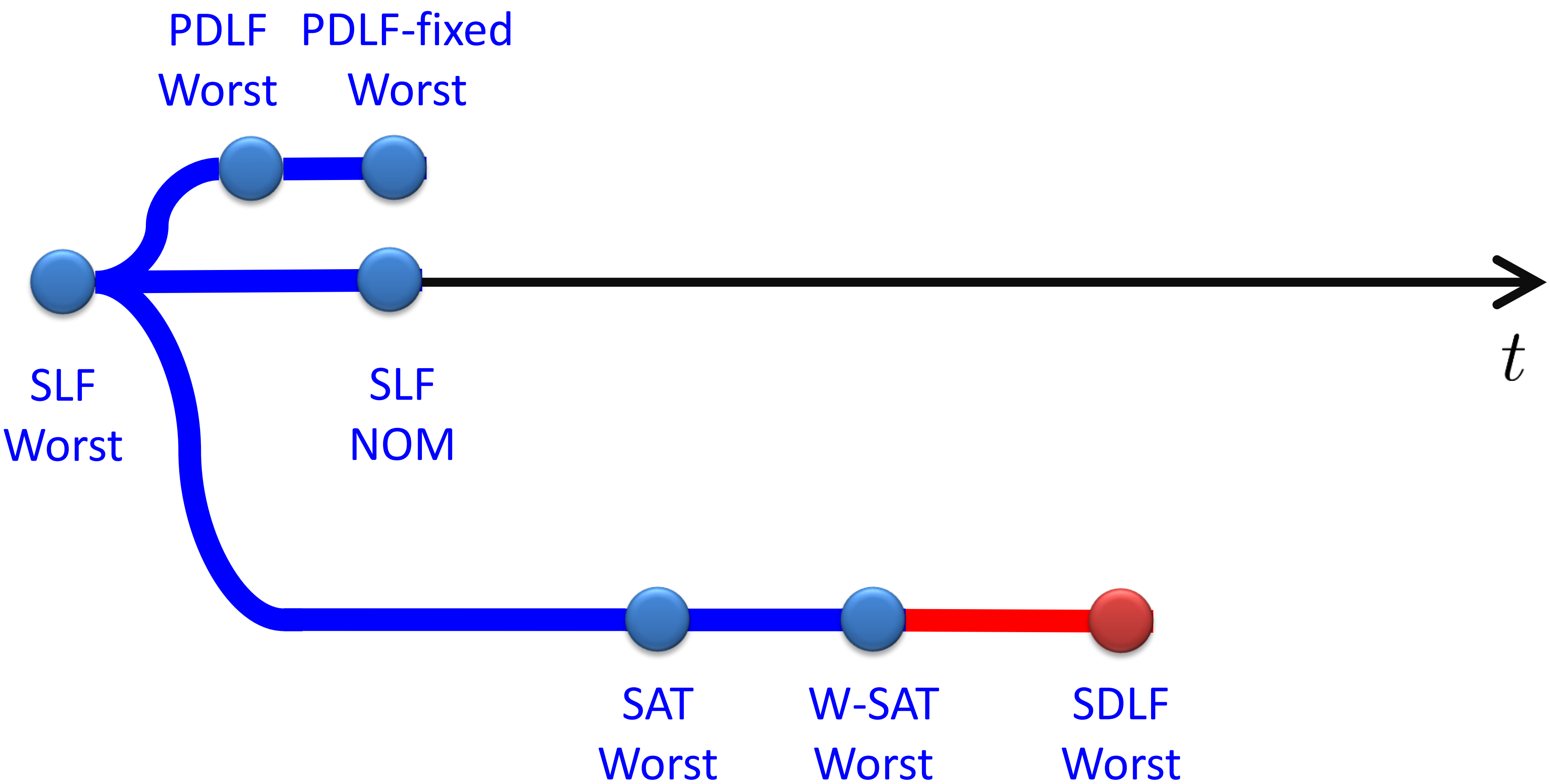
Mao, Automatica, 2003.



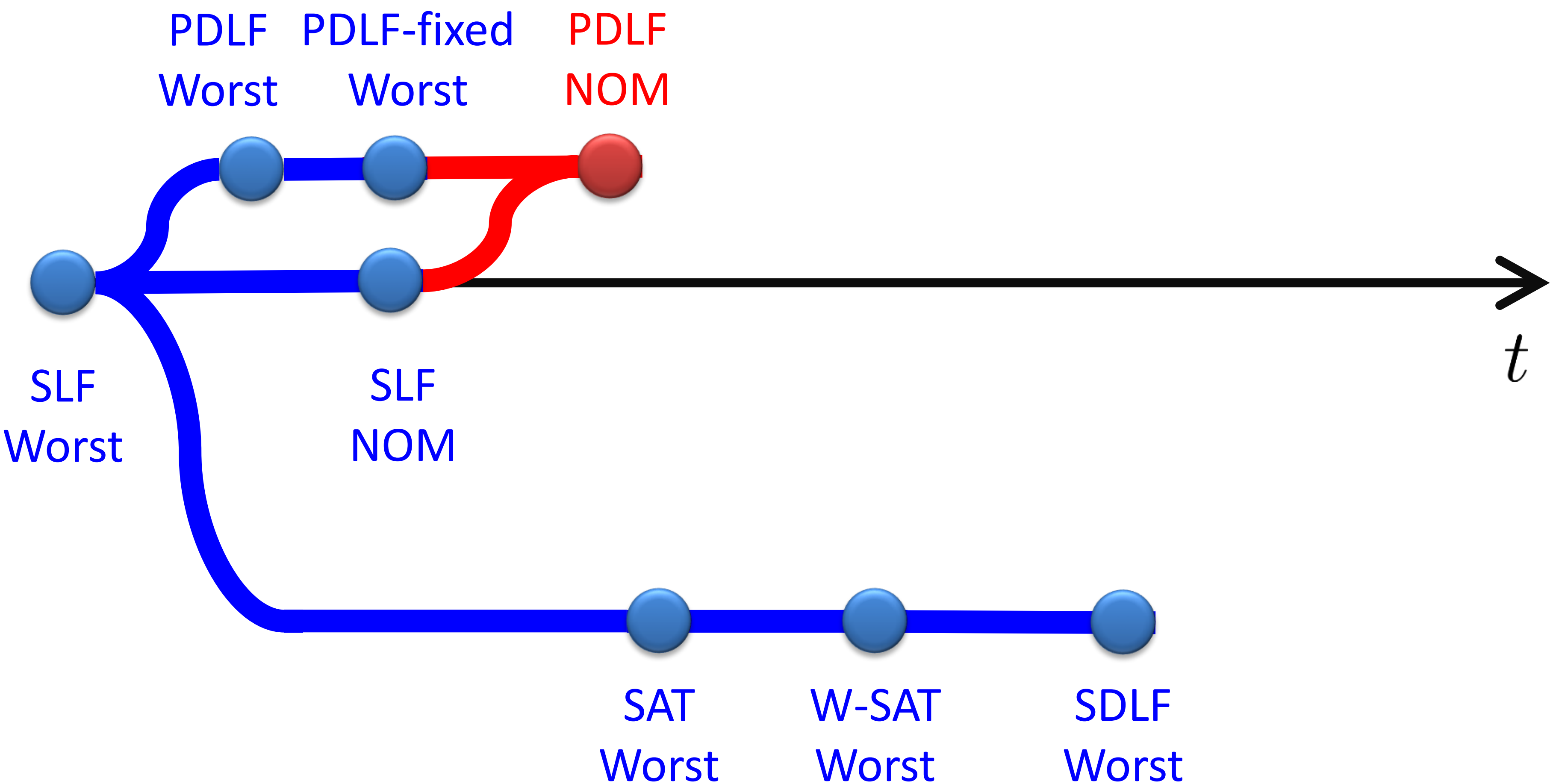
Cao et al., IEE Systems & Control Letters, 2005.



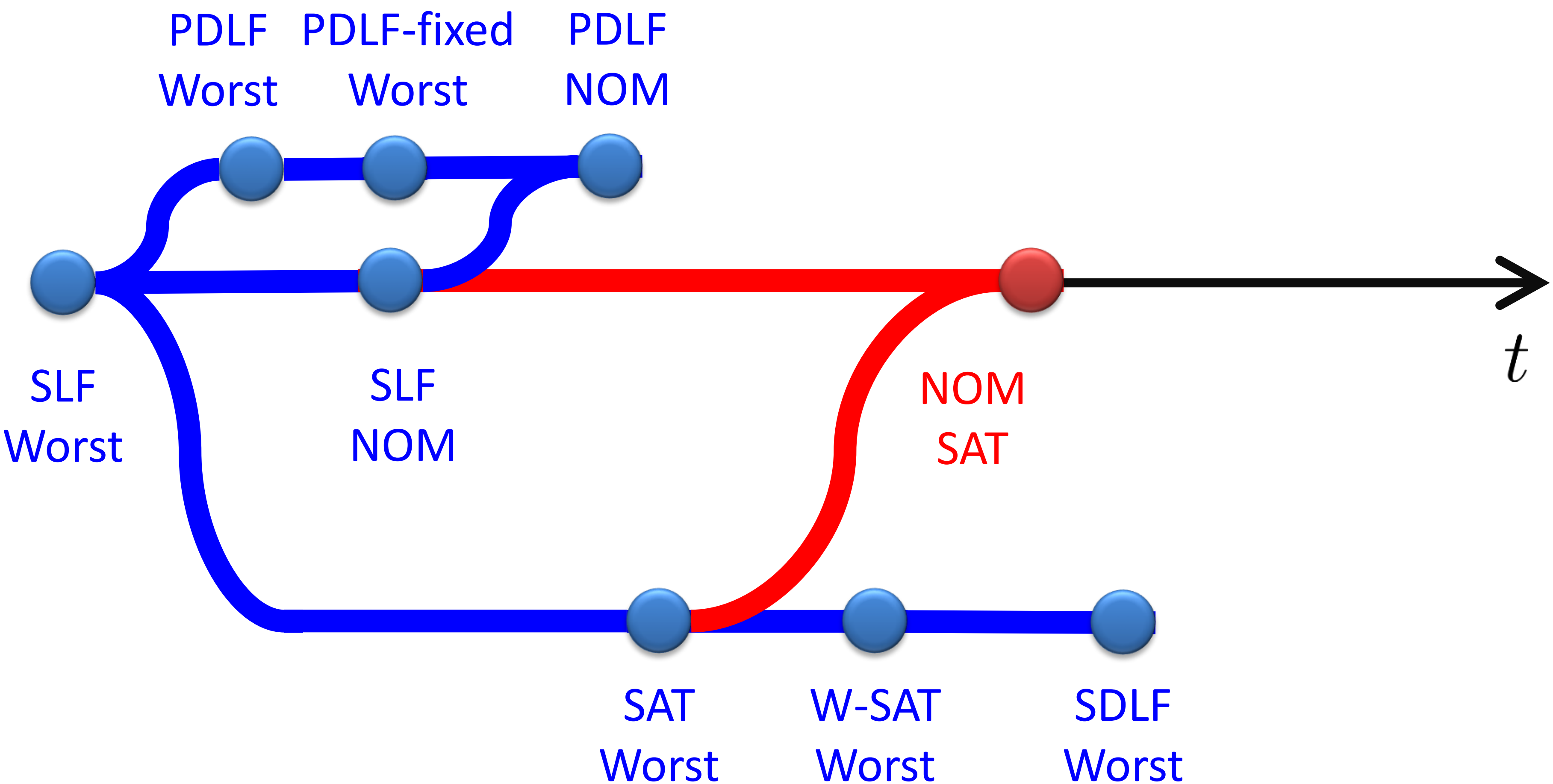
Huang and Cao, Automatica, 2011.



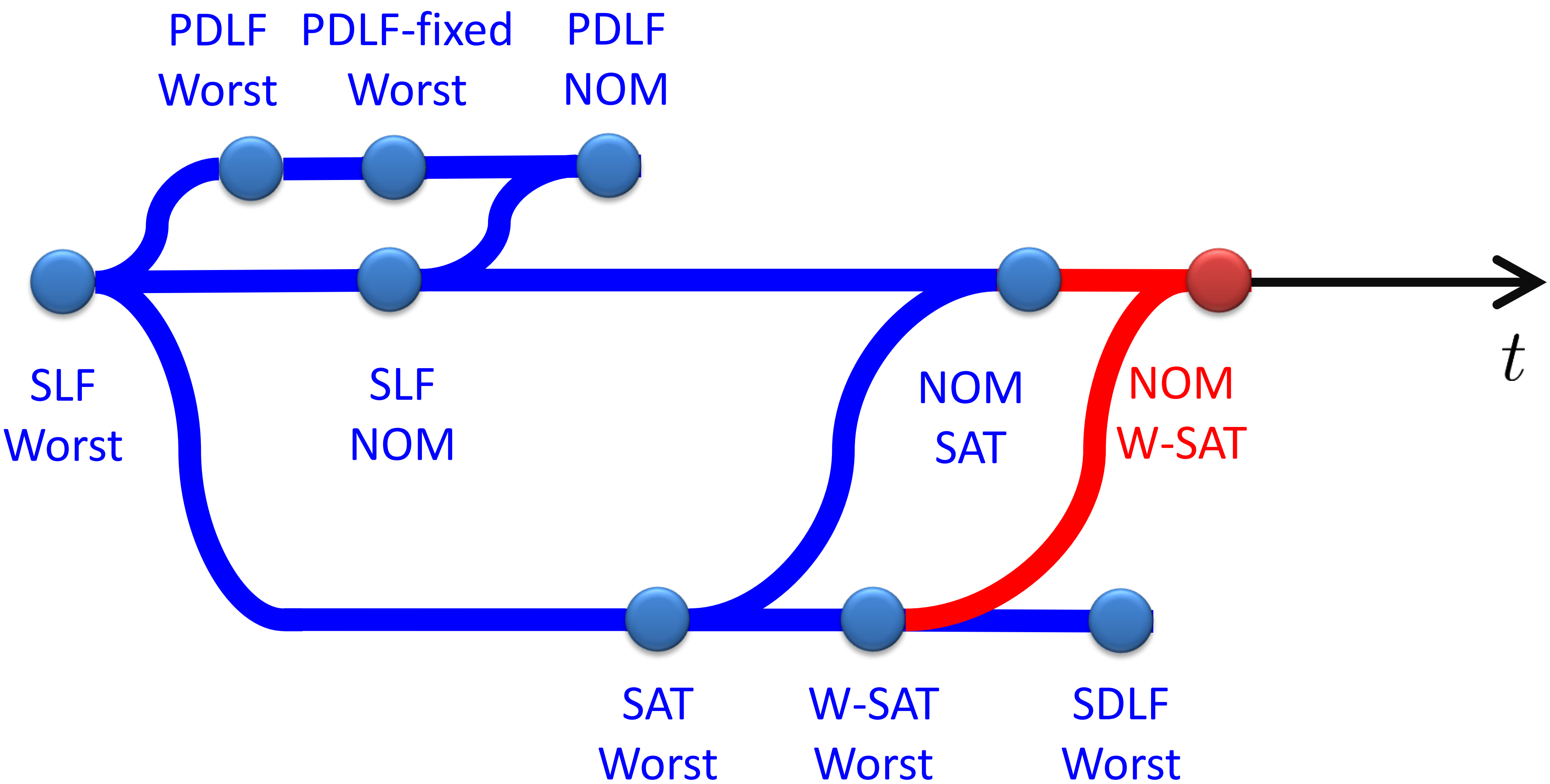
Zhang et al., Automatica, 2013.



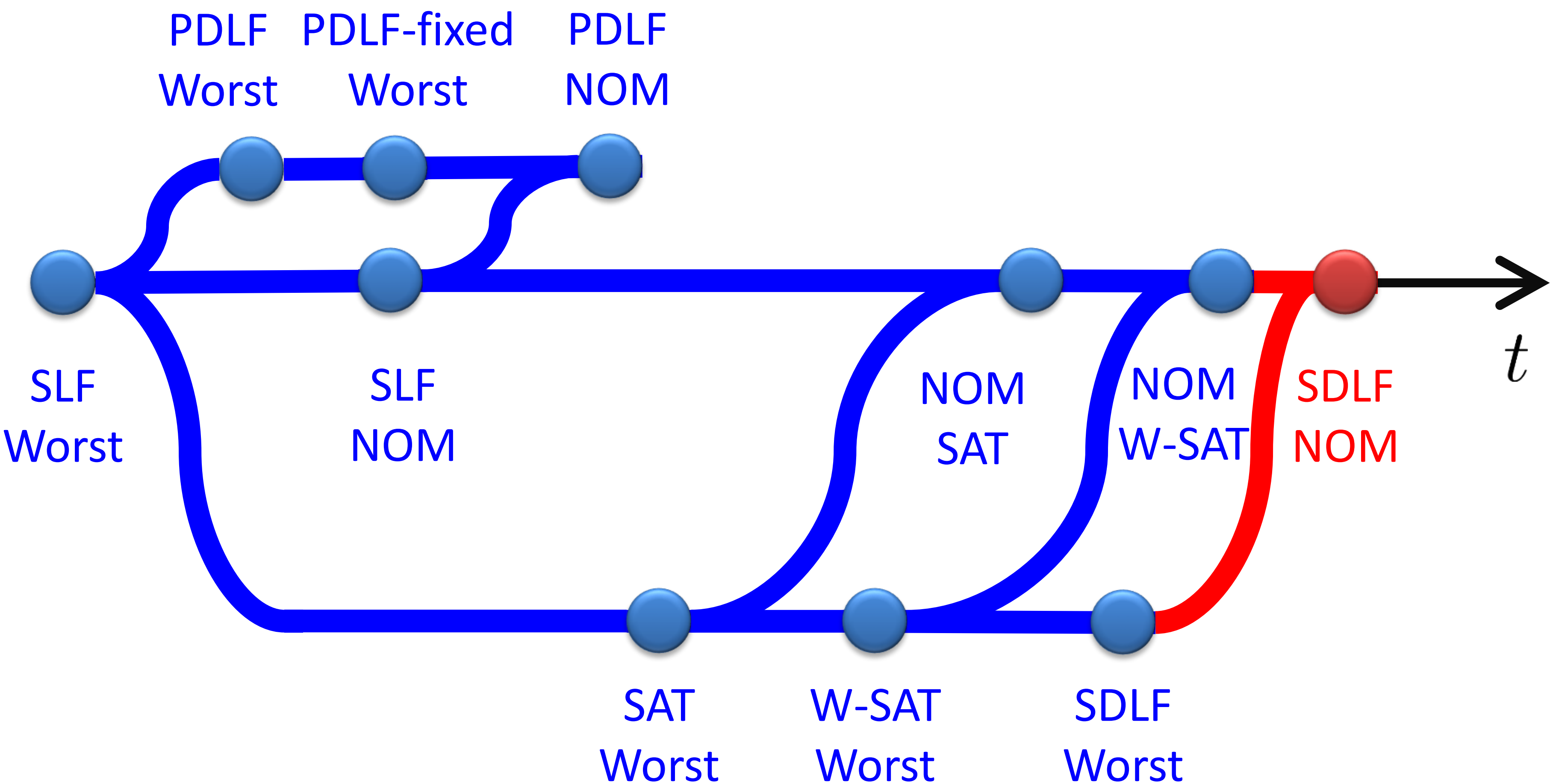
Ding et al., Automatica, 2004.



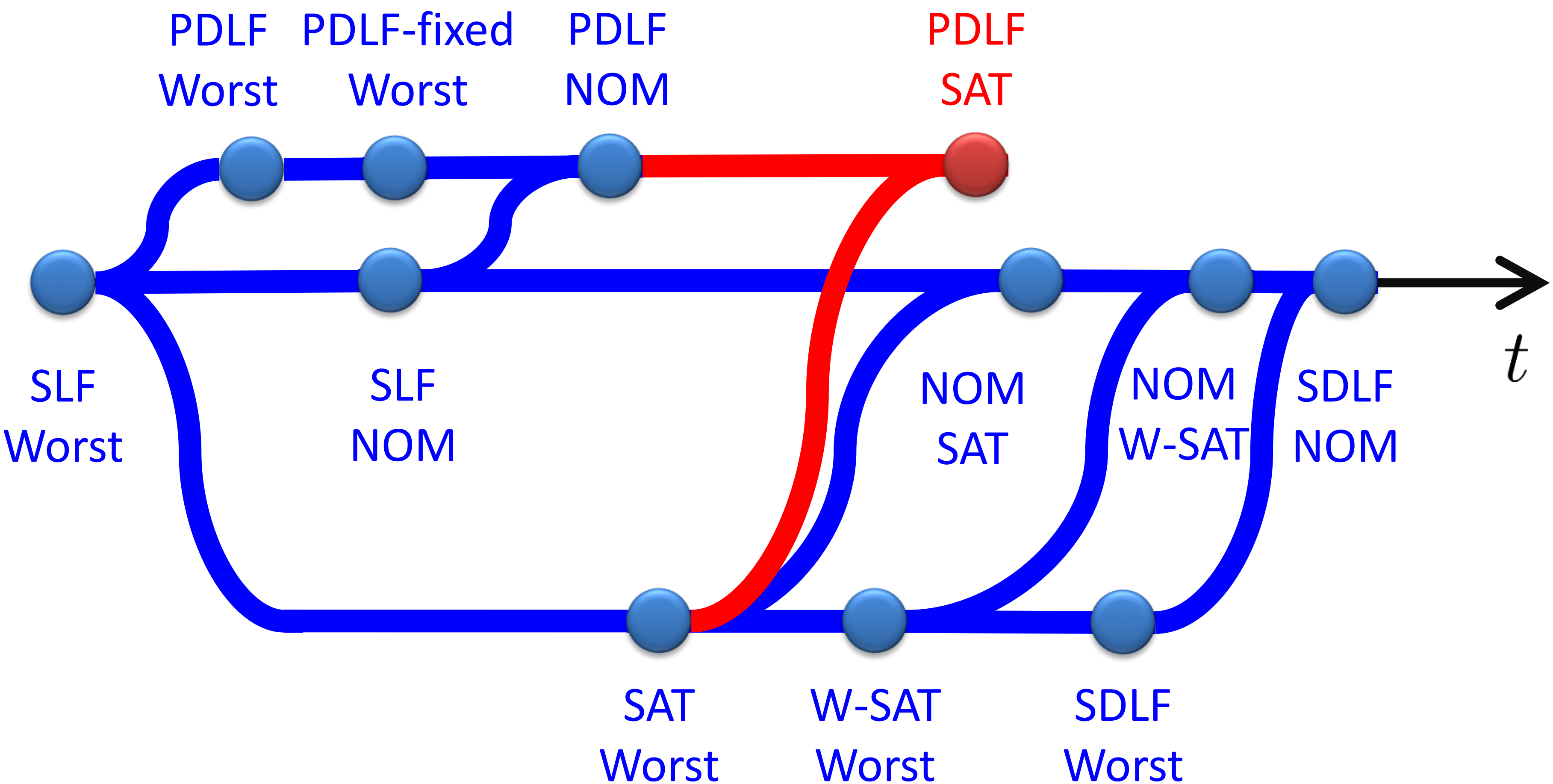
Oravec and Bakošová, IFAC ROCOND, 2015.



Oravec and Bakošová, IFAC ROCOND, 2015.



Oravec and Bakošová, IFAC ROCOND, 2015.



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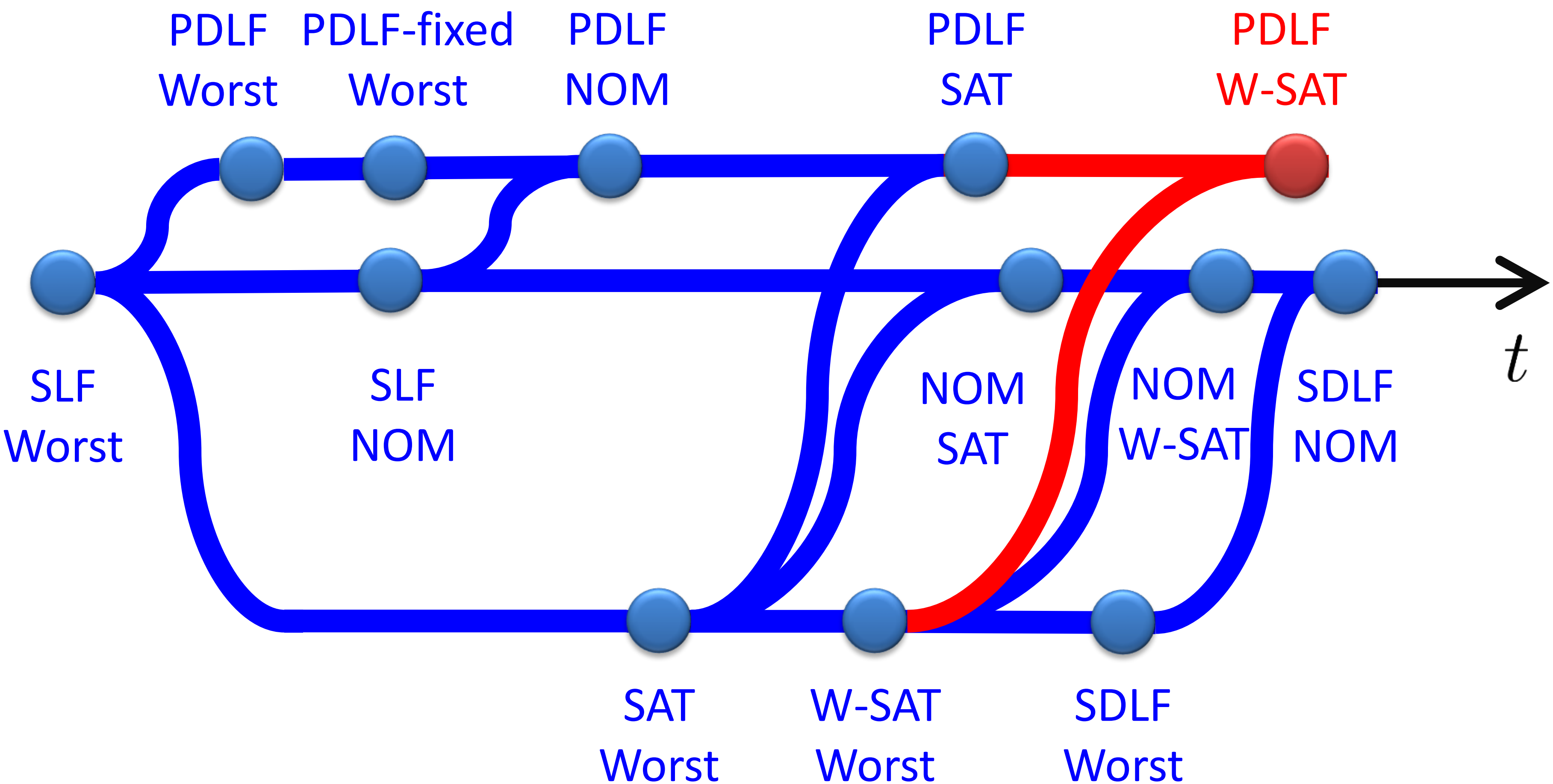
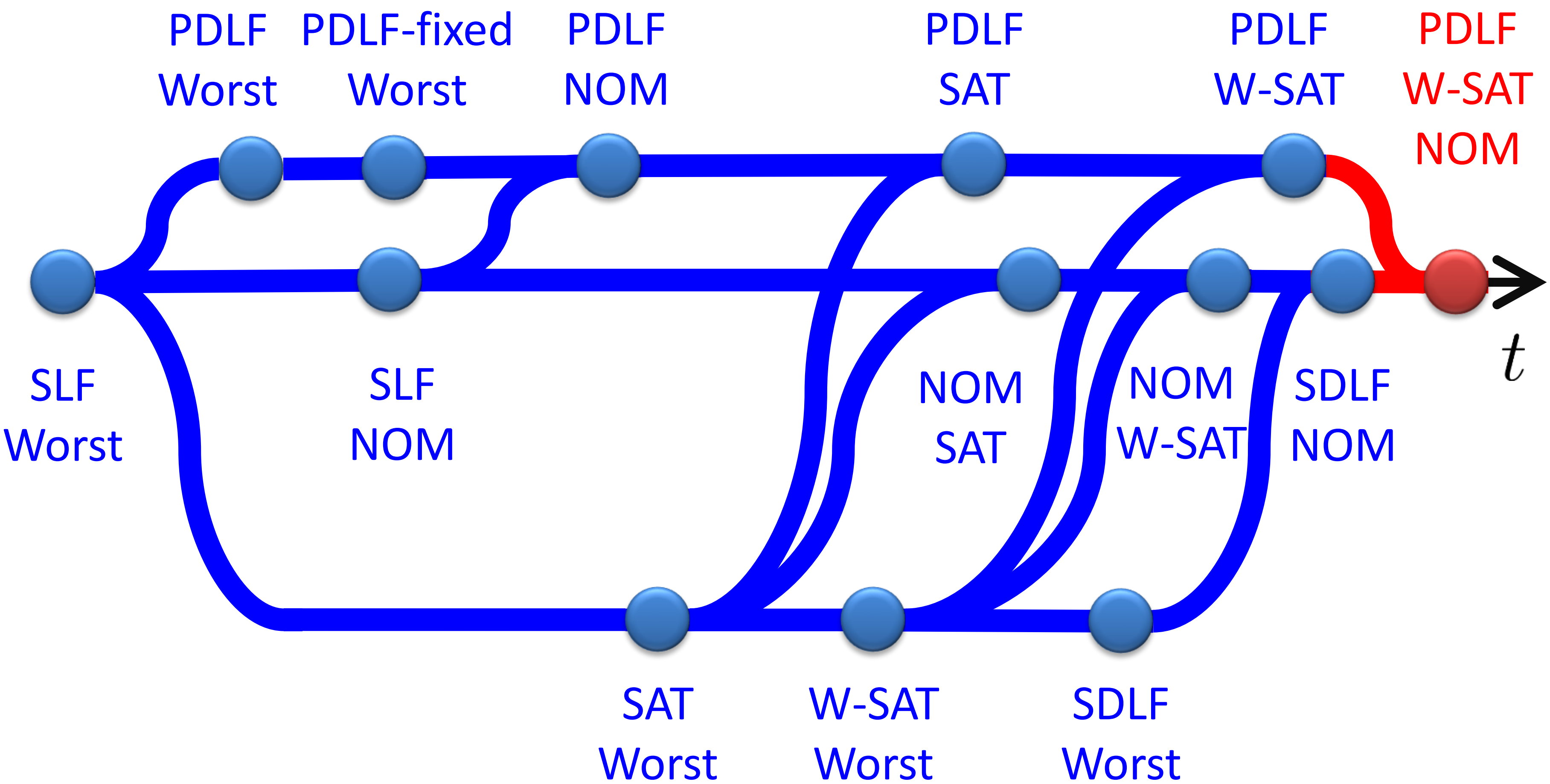
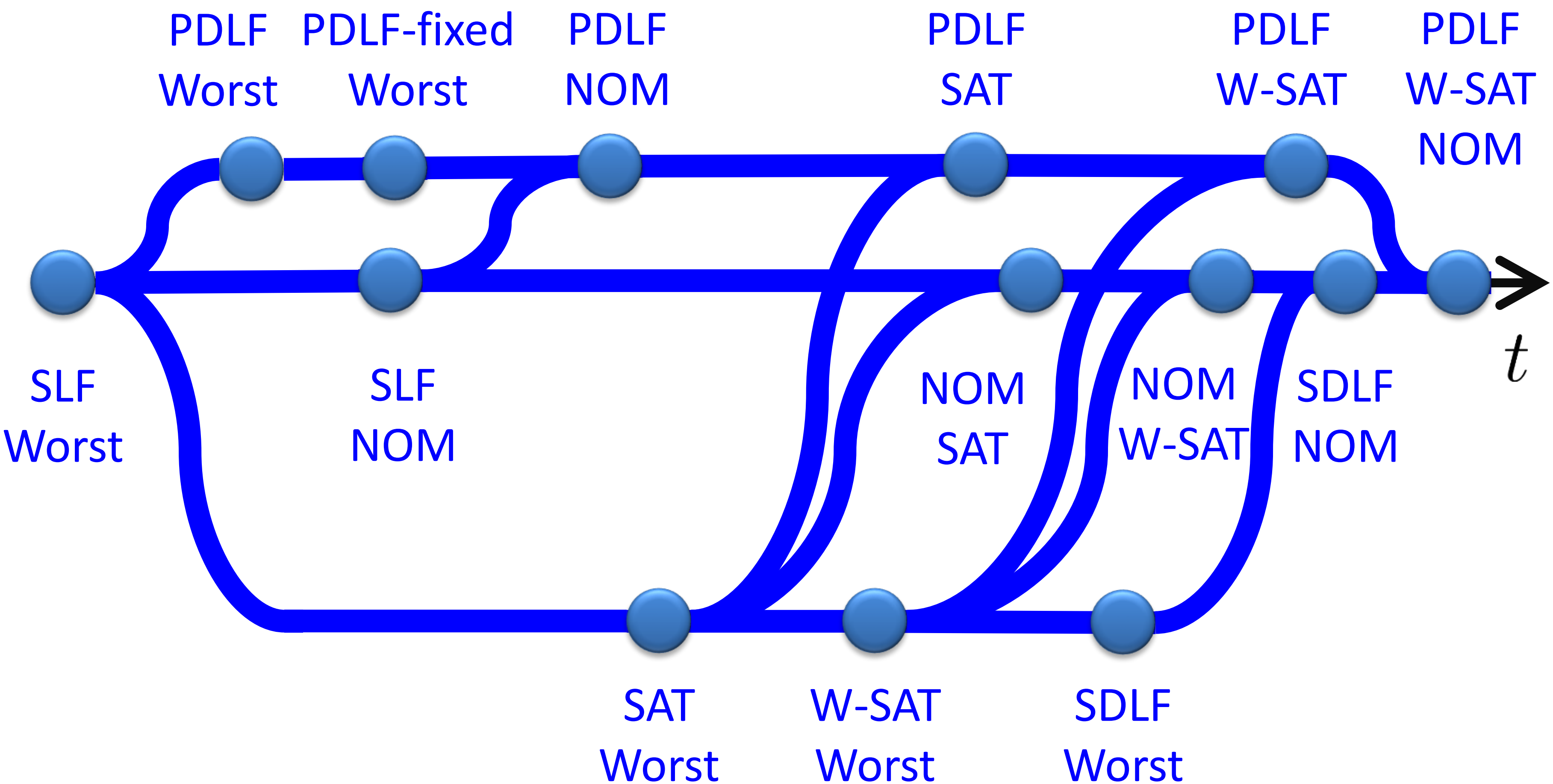


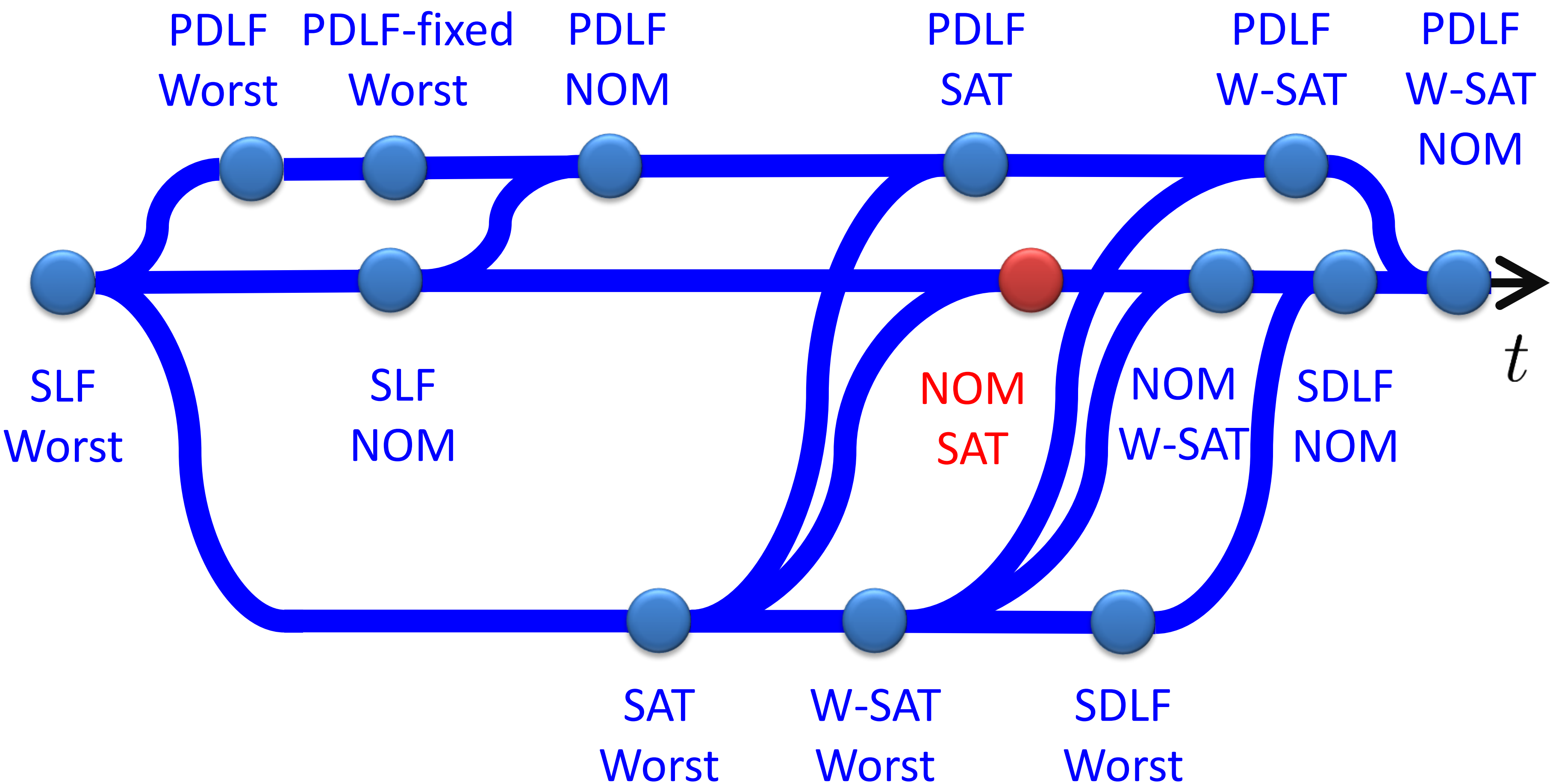
Oravec and Bakošová, IFAC ROCOND, 2015.



Oravec and Bakošová, IFAC ROCOND, 2015.



Oravec and Bakošová, IFAC ROCOND, 2015.



Oravec and Bakošová, IFAC ROCOND, 2015.

NOM & SAT

$RMPC_1$ + $RMPC_2$ = $RMPC_{A1}$

NOM & SAT

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Single LF

Nominal

Bound

[1] *Wan and Kothare, Automatica,* 2003.

NOM & SAT

$RMPC_1$	+	$RMPC_2$	=	$RMPC_{A1}$
Single LF		Single LF		
Nominal		Worst Case		
Bound		Saturation		

[1] *Wan and Kothare, Automatica,* 2003.

[2] *Cao and Lin, Control Theory and Applications IEE Proc.,* 2005.

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Single LF		Single LF		Single LF
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Lyapunov Function

Lyapunov function:

$$V(x(k)) = x(k)^\top P x(k)$$

Lyapunov matrix:

$$0 \prec P = P^\top \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}, \; P = \gamma X^{-1}$$

Stability condition:

$$\Delta V(x_\mathrm{nom}(k)) \leq -J(x_\mathrm{nom}(k))$$

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Lyapunov Function

$$\begin{bmatrix} X & \star & \star & \star \\ A_0 X + B_0(E^j Z + \tilde{E}^j Y) & X & \star & \star \\ \sqrt{Q}X & 0 & \gamma I & \star \\ \sqrt{R}(E^j Z + \tilde{E}^j Y) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0$$

Stability condition:

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Stability condition:

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Lyapunov Function

$$\begin{bmatrix} X & \star \\ A_v X + B_v(E^j Z + \tilde{E}^j Y) & X \end{bmatrix} \succ 0$$

$$\forall v \in \{1, 2, \ldots, n_{\mathrm{v}}\}, \quad \forall j \in \{1, 2, \ldots, 2^{n_{\mathrm{u}}}\}$$

Stability condition:

$$\Delta V(x(k)) < 0, \qquad \forall x \in \mathbb{X}$$

Lyapunov Function

$$\begin{bmatrix} X & \star & \star & \star \\ A_0 X + B_0 (E^j Z + \tilde{E}^j Y) & X & \star & \star \\ \sqrt{Q} X & 0 & \gamma I & \star \\ \sqrt{R} (E^j Z + \tilde{E}^j Y) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0$$

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Feedback Law

Feedback law:

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Feedback law parametrization:

$$F(k) = Z(k)\, X^{-1}(k)$$

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Feedback law parametrization:

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Control input saturation:

$$u_{\text{sat}}(k) = \begin{cases} u_{\max} & \text{if } u(k) \geq u_{\max}, \\ -u_{\max} & \text{if } u(k) \leq -u_{\max}, \\ u(k) & \text{otherwise.} \end{cases}$$

Feedback Law

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Feedback Law

$$\begin{bmatrix} X & \star & \star & \star \\ A_0 X + B_0 (E^j Z + \tilde{E}^j Y) & X & \star & \star \\ \sqrt{Q} X & 0 & \gamma I & \star \\ \sqrt{R}(E^j Z + \tilde{E}^j Y) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X & \star \\ A_v X + B_v (E^j Z + \tilde{E}^j Y) & X \end{bmatrix} \succ 0$$

$$\forall v \in \{1, 2, \ldots, n_{\mathrm{v}}\}, \;\; \forall j \in \{1, 2, \ldots, 2^{n_{\mathrm{u}}}\}.$$

Robust Invariant Ellipsoid

$$\varepsilon = \{x \mid x(k)^\top P x(k) \leq \gamma\}, \quad P = \gamma X^{-1}$$

$$\begin{bmatrix} 1 & \star \\ x(k) & X \end{bmatrix} \succeq 0$$

Robust Invariant Ellipsoid

$$\varepsilon = \{x \,|\, x(k)^\top P x(k) \leq \gamma\}, \quad P = \gamma X^{-1}$$

$$\begin{bmatrix} 1 & \star \\ x(k) & X \end{bmatrix} \succeq 0$$

Objective function:

$$\min \gamma$$

Robust Invariant Ellipsoid

$$\varepsilon = \{x \mid x(k)^\top P x(k) \leq \gamma\}, \quad P = \gamma X^{-1}$$

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Robust Invariant Ellipsoid

$$\varepsilon = \{x \mid x(k)^\top P x(k) \leq \gamma\}, \leftarrow P = \gamma X^{-1}$$

$x(k)$

Objective function:

$\min \gamma$

Robust Invariant Ellipsoid

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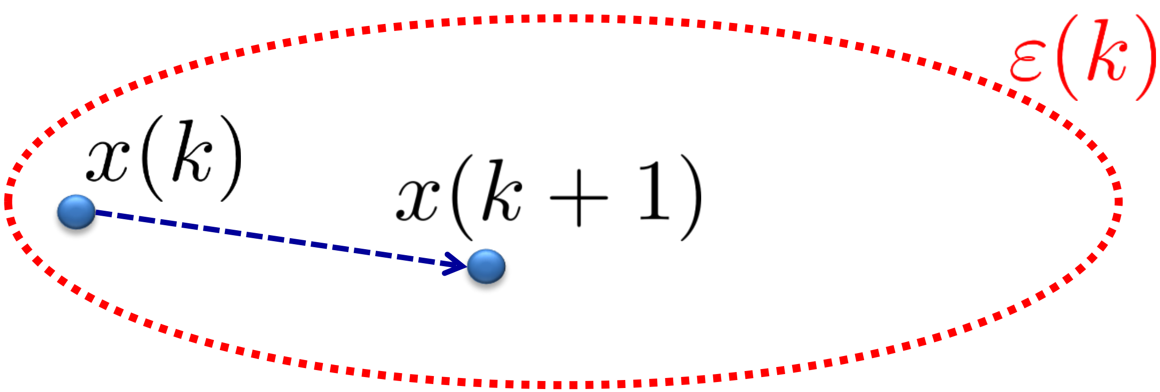

Objective function:

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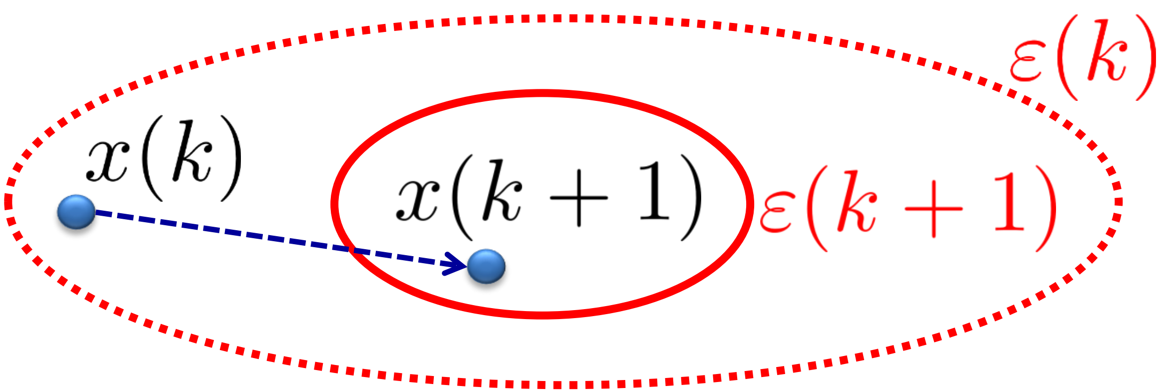


Objective function:

$\min \gamma$

Robust Invariant Ellipsoid

$$\varepsilon = \{x \mid x(k)^\top P x(k) \leq \gamma\}, \quad P = \gamma X^{-1}$$

$$\begin{bmatrix} 1 & \star \\ x(k) & X \end{bmatrix} \succeq 0$$

Objective function:

$$\min \gamma + \mathrm{tr}(X)$$

Input / Output Constraints

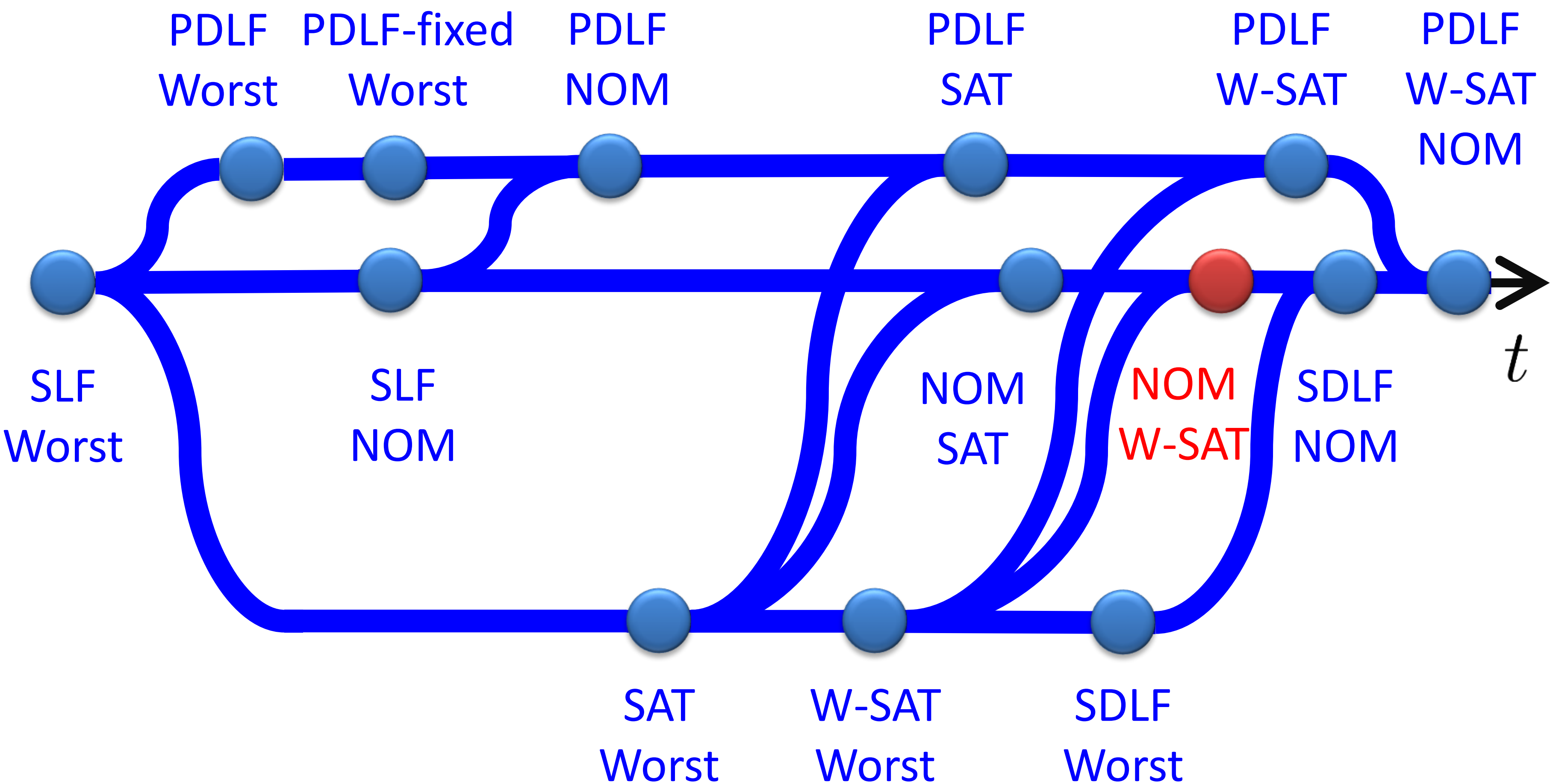
$$\begin{bmatrix} u_{\max}^2 I & \star \\ Z^\top & X \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X & \star \\ C(A_v X + B_v(E^j Z + \tilde{E}^j Y)) & y_{\max}^2 I \end{bmatrix} \succeq 0$$

Input / Output Constraints

$$\begin{bmatrix} u_{\max}^2 I & \star \\ Z^\top & X \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X & \star \\ C(A_v X + B_v(E^j Z + \tilde{E}^j Y)) & y_{\max}^2 I \end{bmatrix} \succeq 0$$



PDLF Worst
PDLF-fixed Worst
PDLF NOM
PDLF SAT
PDLF W-SAT
PDLF W-SAT NOM
SLF Worst
SLF NOM
NOM SAT
NOM W-SAT
SDLF NOM
t
SAT Worst
W-SAT Worst
SDLF Worst

NOM & W-SAT

$RMPC_1$ + $RMPC_3$ = $RMPC_{A2}$

Single LF	Single LF
Nominal	Worst Case
Bound	W-SAT

[1] *Wan and Kothare, Automatica*, 2003.

[3] *Huang et al., Automatica*, 2011.

NOM & W-SAT

$RMPC_1$	+	$RMPC_3$	=	$RMPC_{A2}$
Single LF		Single LF		
Nominal		Worst Case		
Bound		W-SAT		

[1] *Wan and Kothare, Automatica,* 2003.
[3] *Huang et al., Automatica,* 2011.

NOM & W-SAT

$RMPC_1$	+	$RMPC_3$	=	$RMPC_{A2}$
Single LF		Single LF		Single LF
Nominal		Worst Case		Nominal
Bound		W-SAT		W-SAT

---

[1] *Wan and Kothare, Automatica*, 2003.

[3] *Huang et al., Automatica*, 2011.

Feedback Law

Feedback law:

$$u_{\text{sat}}(k) = F(k)\, x(k), \quad u(k) \in \mathbb{U}$$

Feedback law parametrization:

$$F(k) = Y(k)\, X^{-1}(k)$$

Control input saturation:

$$u_{\text{sat}}(k) = \begin{cases} u_{\max} & \text{if } u(k) \geq u_{\max}, \\ -u_{\max} & \text{if } u(k) \leq -u_{\max}, \\ u(k) & \text{otherwise.} \end{cases}$$

Input / Output Constraints

$$\begin{bmatrix} u_{\max}^2 I & \star \\ Z^\top & X \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X & \star \\ C(A_v X + B_v(E^j Z + \tilde{E}^j Y)) & y_{\max}^2 I \end{bmatrix} \succeq 0$$

Lyapunov Function

Lyapunov function:

$$V(x(k)) = x(k)^\top P x(k)$$

Lyapunov matrix:

$$0 \prec P = P^\top \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}},\ P = \gamma X^{-1}$$

Stability condition:

$$\Delta V(x_\mathrm{nom}(k)) \leq -J(x_\mathrm{nom}(k))$$

Lyapunov Function

$$\begin{bmatrix} X & \star & \star & \star \\ A_0 X + B_0 (E^j Z + \tilde{E}^j Y) & X & \star & \star \\ \sqrt{Q} X & 0 & \gamma I & \star \\ \sqrt{R} (E^j Z + \tilde{E}^j Y) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X & \star \\ A_v X + B_v (E^j Z + \tilde{E}^j Y) & X \end{bmatrix} \succ 0$$

$$\forall v \in \{1, 2, \ldots, n_{\mathrm{v}}\}, \;\; \forall j \in \{1, 2, \ldots, 2^{n_{\mathrm{u}}}\}.$$

Lyapunov Function

$$\begin{bmatrix} X & \star & \star & \star \\ A_0 X + B_0 (E^j Z + \tilde{E}^j Y) & X & \star & \star \\ \sqrt{Q} X & 0 & \gamma I & \star \\ \sqrt{R} (E^j Z + \tilde{E}^j Y) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X & \star & \star & \star \\ A_0 X + B_0 Y & X & \star & \star \\ \sqrt{Q} X & 0 & \gamma_{\mathrm{s}} I & \star \\ \sqrt{R} Y & 0 & 0 & \gamma_{\mathrm{s}} I \end{bmatrix} \succeq 0$$

Lyapunov Function

$$\begin{bmatrix} X & \star & \star & \star \\ A_0X + B_0(E^jZ + \tilde{E}^jY) & X & \star & \star \\ \sqrt{Q}X & 0 & \gamma I & \star \\ \sqrt{R}(E^jZ + \tilde{E}^jY) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X & \star & \star & \star \\ A_0X + B_0Y & X & \star & \star \\ \sqrt{Q}X & 0 & \gamma_{\mathrm{s}} I & \star \\ \sqrt{R}Y & 0 & 0 & \gamma_{\mathrm{s}} I \end{bmatrix} \succeq 0$$

Robust Invariant Ellipsoid

$$\varepsilon = \{x \mid x(k)^\top P x(k) \leq \gamma\}, \quad P = \gamma X^{-1}$$

$$\begin{bmatrix} 1 & \star \\ x(k) & X \end{bmatrix} \succeq 0$$

Objective function:

$$\min \gamma + \beta \gamma_{\mathrm{s}}$$

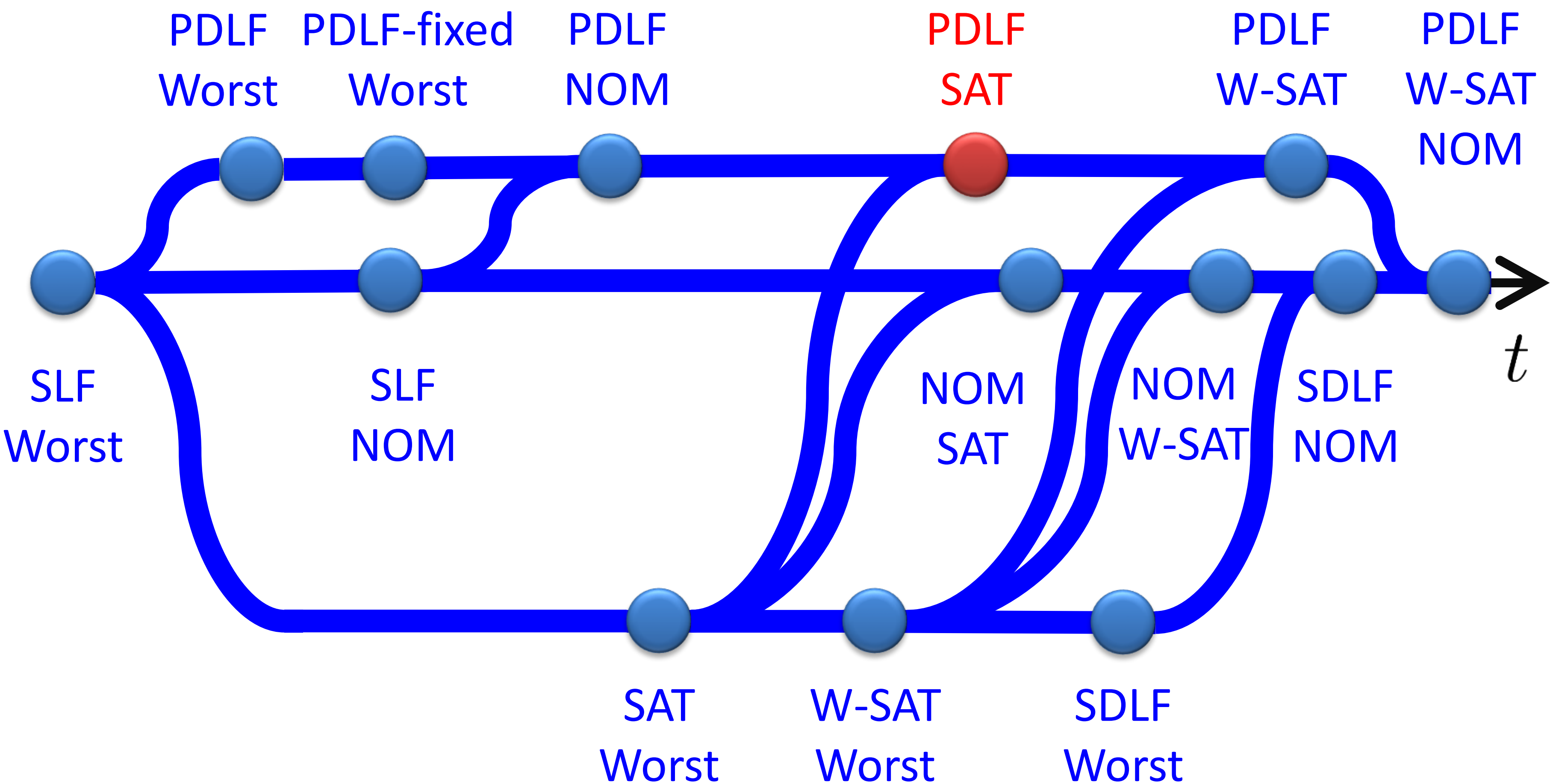
Robust Invariant Ellipsoid

$$\varepsilon = \{x \mid x(k)^\top P x(k) \leq \gamma\}, \quad P = \gamma X^{-1}$$

$$\begin{bmatrix} 1 & \star \\ x(k) & X \end{bmatrix} \succeq 0$$

Objective function:

$$\min \gamma + \beta \gamma_{\mathrm{s}} + \mathrm{tr}(X)$$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PDLF & SAT

$RMPC_2$ + $RMPC_4$ = $RMPC_{A3}$

$RMPC_2$	$RMPC_4$
Single LF	PDLF
Worst Case	Worst Case
ACIS	Bound

[2] *Cao and Lin, Control Theory and Applications IEE Proc.,* 2005.

[4] *Mao, Automatica,* 2003.

PDLF & SAT

$RMPC_2$	+	$RMPC_4$	=	$RMPC_{A3}$
Single LF		PDLF		
Worst Case		Worst Case		
Saturation		Bound		

[2] *Cao and Lin, Control Theory and Applications IEE Proc.*, 2005.

[4] *Mao, Automatica*, 2003.

PDLF & SAT

$RMPC_2$	+	$RMPC_4$	=	$RMPC_{A3}$
Single LF		PDLF		PDLF
Worst Case		Worst Case		Worst Case
Saturation		Bound		Saturation

[2] *Cao and Lin, Control Theory and Applications IEE Proc.,* 2005.

[4] *Mao, Automatica,* 2003.

Feedback Law

Feedback law:

$$u_{\text{sat}}(k) = F(k)\,x(k), \quad u(k) \in \mathbb{U}$$

Feedback law parametrization:

$$F(k) = Y(k)\,X^{-1}(k)$$

Control input saturation:

$$u_{\text{sat}}(k) = \begin{cases} u_{\max} & \text{if } u(k) \geq u_{\max}, \\ -u_{\max} & \text{if } u(k) \leq -u_{\max}, \\ u(k) & \text{otherwise.} \end{cases}$$

Lyapunov Function

Parameter-dependent Lyapunov matrix:

$$P = P^\top \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}, \quad P = \gamma X^{-1}$$

$$X + X^\top \succ G_v \succ 0$$

Stability condition:

$$\Delta V(x(k)) < -J(x(k)), \quad \forall x \in \mathbb{X}$$

Lyapunov Function

$$\begin{bmatrix} X + X^\top - G_v & \star & \star & \star \\ A_v X + B_v(E^j Z + \tilde{E}^j Y) & G_w & \star & \star \\ \sqrt{Q} X & 0 & \gamma I & \star \\ \sqrt{R}(E^j Z + \tilde{E}^j Y) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0$$

$$\forall v \in \{1, 2, \ldots, n_{\mathrm{v}}\}, \quad \forall w \in \{1, 2, \ldots, n_{\mathrm{v}}\},$$

$$\forall j \in \{1, 2, \ldots, 2^{n_{\mathrm{u}}}\}.$$

Input / Output Constraints

$$\begin{bmatrix} u_{\max}^2 I & \star \\ Y^\top & X + X^\top - G_v \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X + X^\top - G_v & \star \\ C(A_v X + B_v(E^j Z + \tilde{E}^j Y)) & y_{\max}^2 I \end{bmatrix} \succeq 0$$

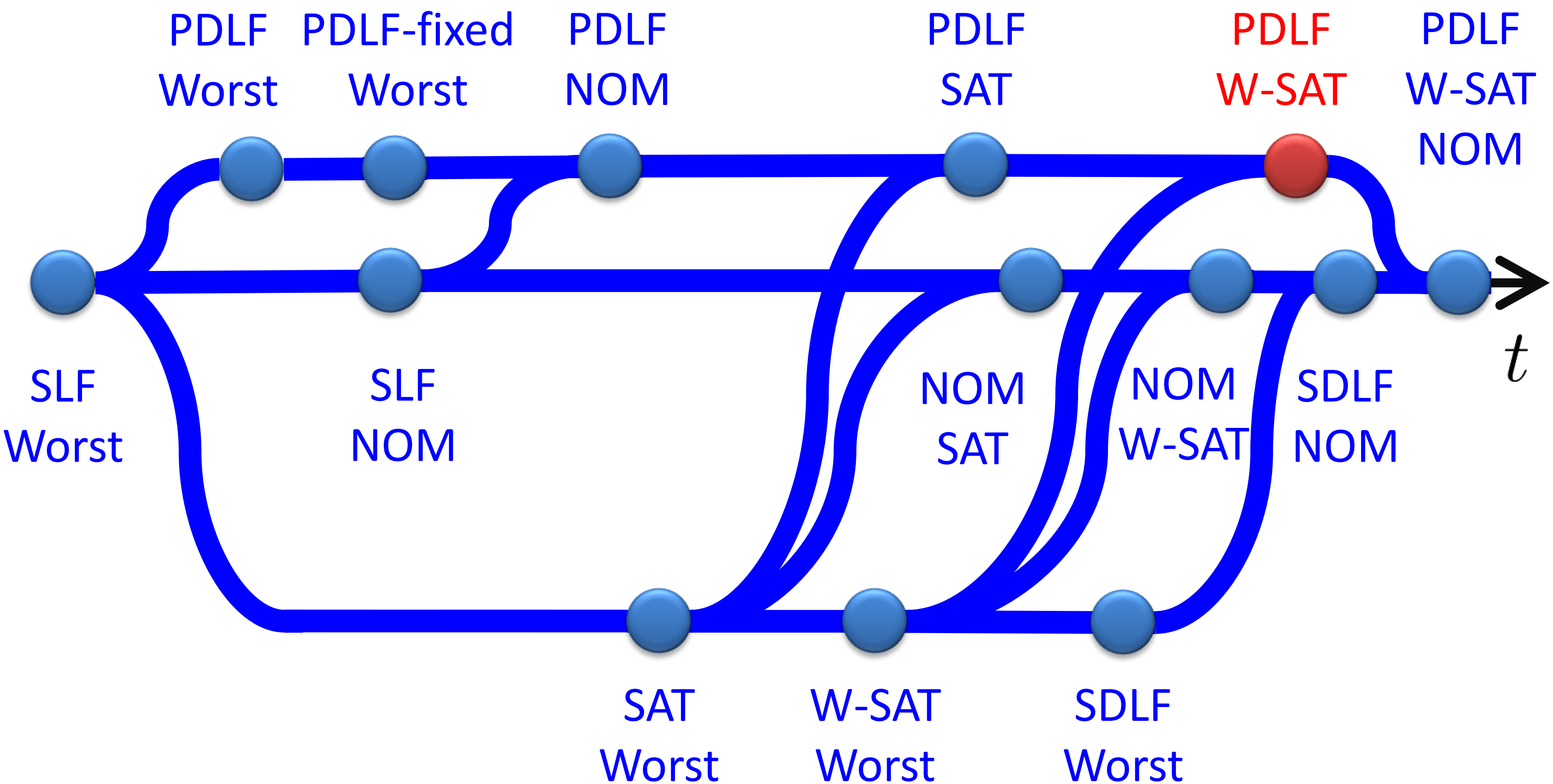
Robust Invariant Ellipsoid

$$\begin{bmatrix} 1 & \star \\ x(k) & G_v \end{bmatrix} \succeq 0$$

$$\forall v \in \{1, 2, \ldots, n_{\mathrm{v}}\}$$

Objective function:

$$\min \gamma + \mathrm{tr}(X)$$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PDLF & W-SAT

$RMPC_3$	+	**$RMPC_4$**	=	**$RMPC_{A4}$**
Single LF		**PDLF**		
Worst Case		**Worst Case**		
W-SAT		**Bound**		

[3] *Huang et al., Automatica,* 2011

[4] *Mao, Automatica,* 2003.

PDLF & W-SAT

$RMPC_3$	+	$RMPC_4$	=	$RMPC_{A4}$
Single LF		PDLF		
Worst Case		Worst Case		
W-SAT		Bound		

[3] *Huang et al., Automatica,* 2011
[4] *Mao, Automatica,* 2003.

PDLF & W-SAT

$RMPC_3$	+	$RMPC_4$	=	$RMPC_{A4}$
Single LF		PDLF		PDLF
Worst Case		Worst Case		Worst Case
W-SAT		Bound		W-SAT

[3] *Huang et al., Automatica,* 2011

[4] *Mao, Automatica,* 2003.

Lyapunov Function

Parameter-dependent Lyapunov matrix:

$$P = P^\top \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}, \quad P = \gamma X^{-1}$$

$$X + X^\top \succ G_v \succ 0$$

Stability condition:

$$\Delta V(x_\mathrm{worst}(k)) \leq -J(x(k))$$

Lyapunov Function

$$\begin{bmatrix} X + X^\top - G_v & \star & \star & \star \\ A_v X + B_v(E^j Z + \tilde{E}^j Y) & G_w & \star & \star \\ \sqrt{Q} X & 0 & \gamma I & \star \\ \sqrt{R}(E^j Z + \tilde{E}^j Y) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0$$

$$\forall v \in \{1, 2, \ldots, n_{\mathrm{v}}\}, \quad \forall w \in \{1, 2, \ldots, n_{\mathrm{v}}\},$$

$$\forall j \in \{1, 2, \ldots, 2^{n_{\mathrm{u}}}\}.$$

Lyapunov Function

$$\begin{bmatrix} X + X^\top - G_v & \star & \star & \star \\ A_v X + B_v (E^j Z + \tilde{E}^j Y) & G_w & \star & \star \\ \sqrt{Q} X & 0 & \gamma I & \star \\ \sqrt{R} (E^j Z + \tilde{E}^j Y) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X + X^\top - G_v & \star & \star & \star \\ A_v X + B_v Y & G_w & \star & \star \\ \sqrt{Q} X & 0 & \gamma_{\mathrm{s}} I & \star \\ \sqrt{R} Y & 0 & 0 & \gamma_{\mathrm{s}} I \end{bmatrix} \succeq 0$$

Lyapunov Function

$$\begin{bmatrix} X + X^\top - G_v & \star & \star & \star \\ A_v X + B_v(E^j Z + \tilde{E}^j Y) & G_w & \star & \star \\ \sqrt{Q} X & 0 & \gamma I & \star \\ \sqrt{R}(E^j Z + \tilde{E}^j Y) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X + X^\top - G_v & \star & \star & \star \\ A_v X + B_v Y & G_w & \star & \star \\ \sqrt{Q} X & 0 & \gamma_\mathrm{s} I & \star \\ \sqrt{R} Y & 0 & 0 & \gamma_\mathrm{s} I \end{bmatrix} \succeq 0$$

Input / Output Constraints

$$\begin{bmatrix} u_{\max}^2 I & \star \\ Y^\top & X + X^\top - G_v \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X + X^\top - G_v & \star \\ C(A_v X + B_v(E^j Z + \tilde{E}^j Y)) & y_{\max}^2 I \end{bmatrix} \succeq 0$$

Robust Invariant Ellipsoid

$$\begin{bmatrix} 1 & \star \\ x(k) & G_v \end{bmatrix} \succeq 0$$

$$\forall v \in \{1, 2, \ldots, n_{\mathrm{v}}\}$$

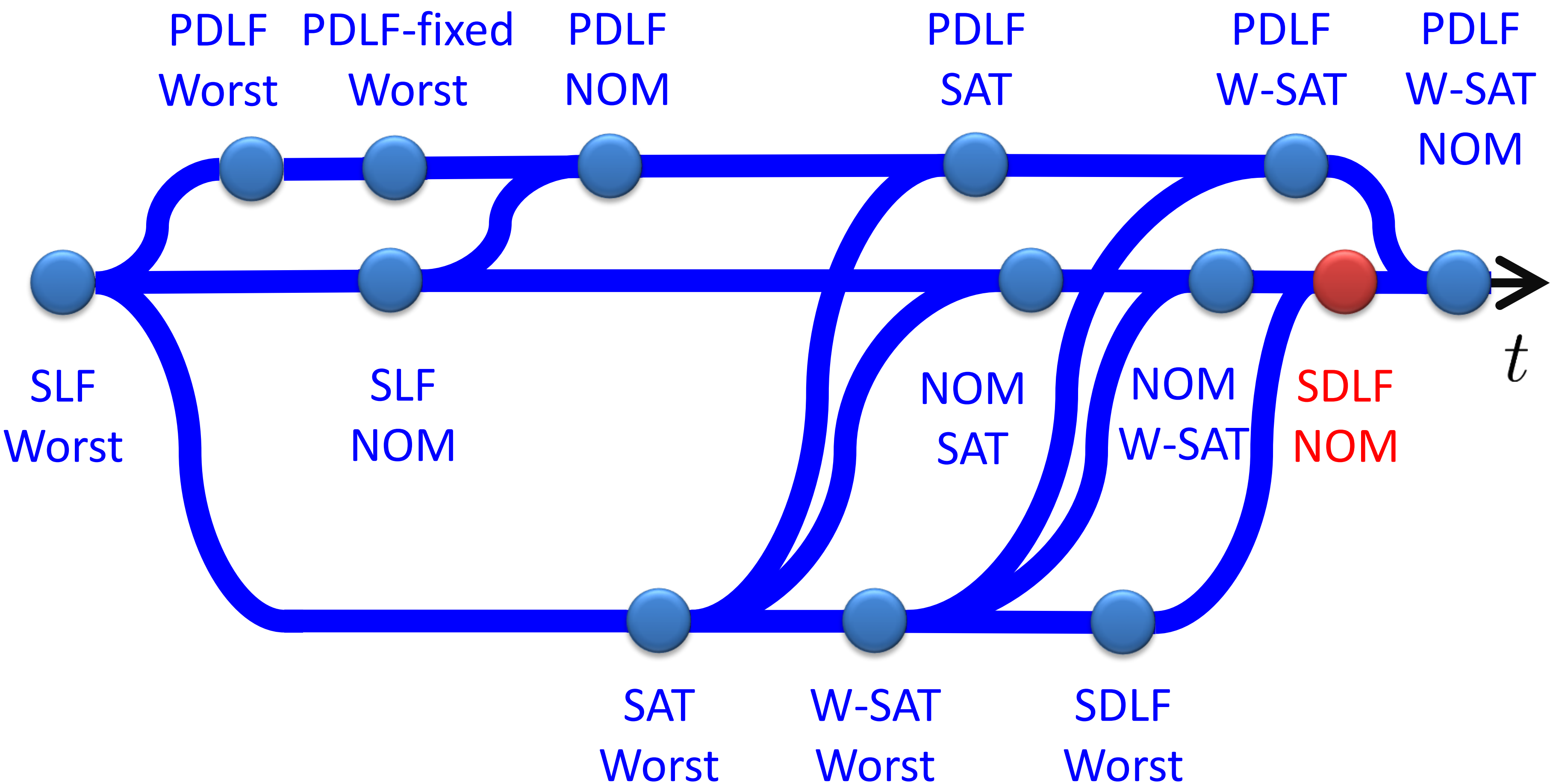
Robust Invariant Ellipsoid

$$\begin{bmatrix} 1 & \star \\ x(k) & G_v \end{bmatrix} \succeq 0$$

$$\forall v \in \{1, 2, \ldots, n_{\mathrm{v}}\}$$

Objective function:

$$\min \gamma + \beta \, \gamma_{\mathrm{s}}$$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NSO & SDLF

$RMPC_1$	+	$RMPC_5$	=	$RMPC_{A5}$
Single LF		SDLF		
Nominal		Worst Case		
Bound		Saturation		

[1] *Wan and Kothare, Automatica.,* 2003

[5] *Zhang et al., Automatica,* 2013.

NSO & SDLF

$RMPC_1$	+	$RMPC_5$	=	$RMPC_{A5}$
Single LF		SDLF		
Nominal		Worst Case		
Bound		Saturation		

[1] *Wan and Kothare, Automatica.*, 2003

[5] *Zhang et al., Automatica*, 2013.

NSO & SDLF

$RMPC_1$	+	$RMPC_5$	=	$RMPC_{A5}$
Single LF		SDLF		SDLF
Nominal		Worst Case		Nominal
Bound		Saturation		Saturation

[1] *Wan and Kothare, Automatica.,* 2003

[5] *Zhang et al., Automatica,* 2013.

Lyapunov Function

Parameter-dependent Lyapunov matrix:

$$P = P^\top \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}, \quad P = \gamma X^{-1}$$

$$X + X^\top \succ S_v \succ 0$$

Stability condition:

$$\Delta V(x_\mathrm{nom}(k)) \leq -J(x(k))$$

Feedback Law

Feedback law:

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Lyapunov Function

$$\begin{bmatrix} X + X^\top - S_v & \star & \star & \star \\ A_0 X + B_0(E^j Z + \tilde{E}^j Y) & S_w & \star & \star \\ \sqrt{Q} X & 0 & \gamma I & \star \\ \sqrt{R}(E^j Z + \tilde{E}^j Y) & 0 & 0 & \gamma I \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X + X^\top - S_v & \star \\ A_v X + B_v(E^j Z + \tilde{E}^j Y) & S_w \end{bmatrix} \succ 0$$

$$\forall v \in \{1, 2, \ldots, n_\mathrm{v}\}, \;\; \forall w \in \{1, 2, \ldots, n_\mathrm{v}\},$$

$$\forall j \in \{1, 2, \ldots, 2^{n_\mathrm{u}}\}.$$

Input / Output Constraints

$$\begin{bmatrix} u_{\max}^2 I & \star \\ Z^\top & S_v \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X + X^\top - S_v & \star \\ C(A_v X + B_v(E^j Z + \tilde{E}^j Y)) & y_{\max}^2 I \end{bmatrix} \succeq 0$$

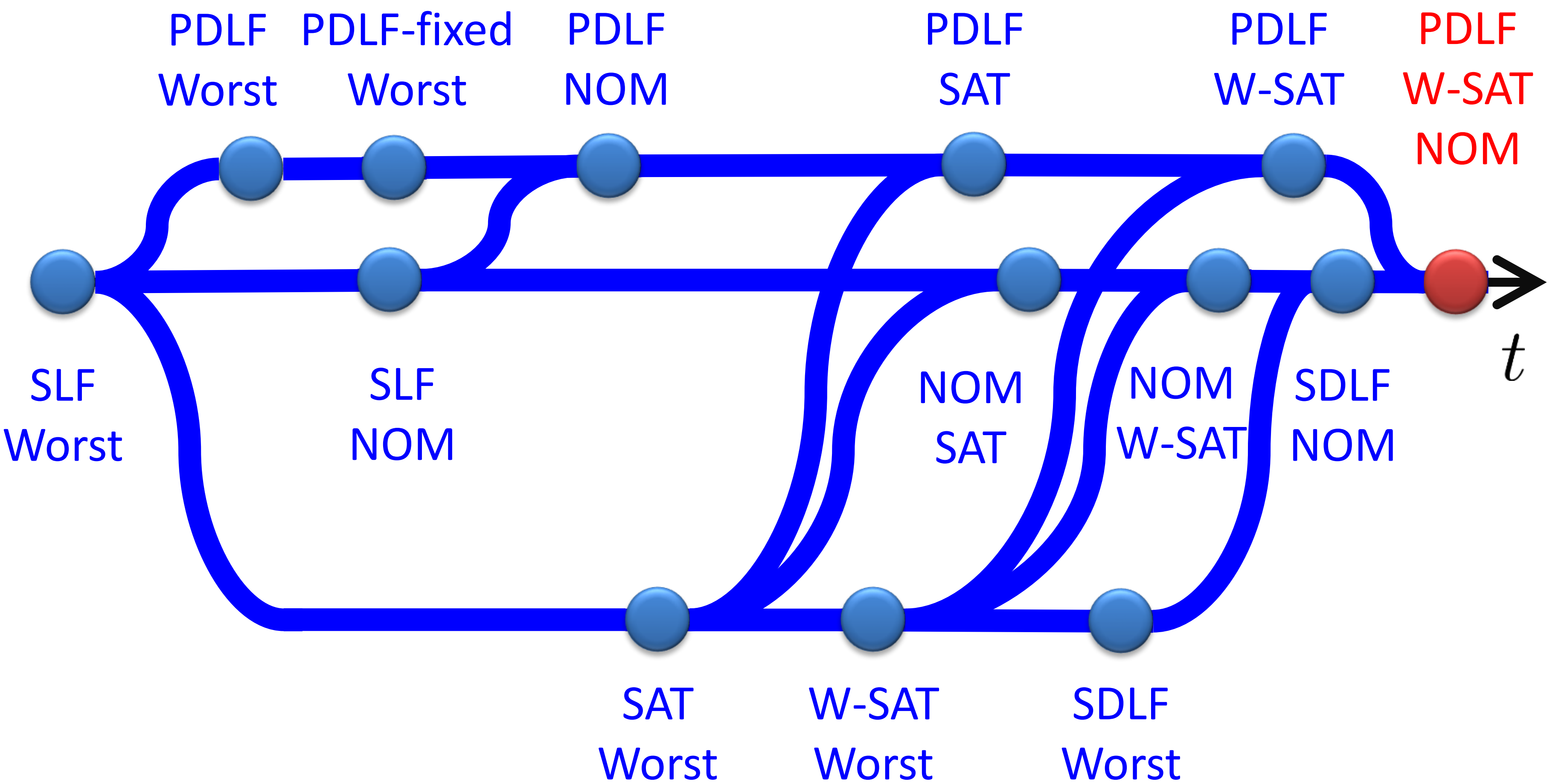
Robust Invariant Ellipsoid

$$\varepsilon = \{x \,|\, x(k)^\top P x(k) \leq \gamma\}, \quad P = \gamma X^{-1}$$

$$\begin{bmatrix} 1 & \star \\ x(k) & X + X^\top - S_v \end{bmatrix} \succeq 0$$

Objective function:

$$\min \gamma + \mathrm{tr}(X)$$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SDLF
Worst

PDLF & SAT & NOM

$RMPC_{A7}$

$NOM_{[1]}$

$SAT_{[2]}$

$PDLF_{[4]}$

[1] *Wan and Kothare, Automatica.,* 2003

[2] *Cao and Lin, Control Theory and Applications IEE Proc., 2005.*

[4] *Mao, Automatica, 2003.*

PDLF & W-SAT & NOM

$RMPC_{A8}$

$NOM_{[1]}$

$W\text{-}SAT_{[3]}$

$PDLF_{[4]}$

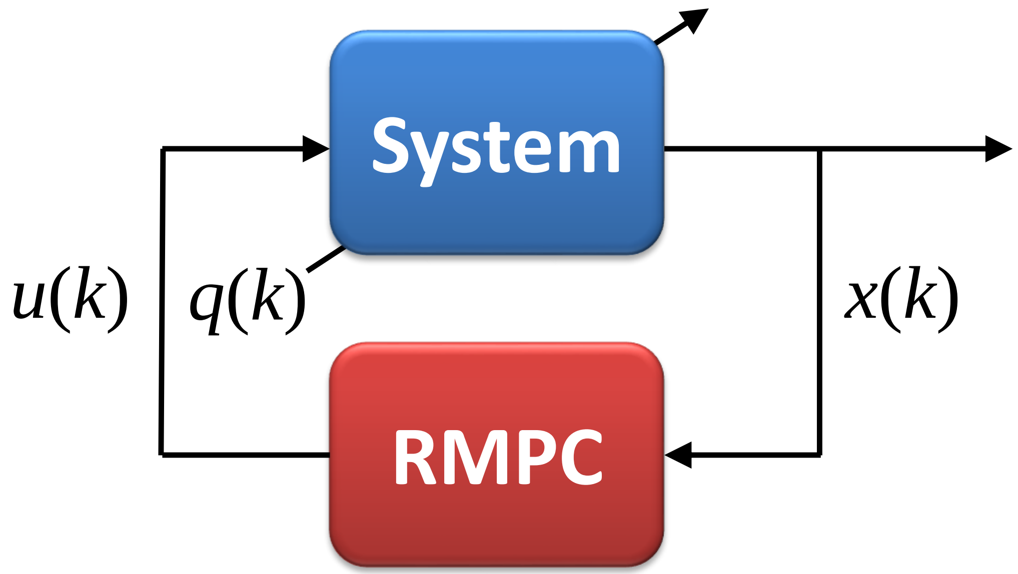
[1] *Wan and Kothare, Automatica.,* 2003

[3] *Huang et al., Automatica, 2011.*

[4] *Mao, Automatica, 2003.*

Case Study

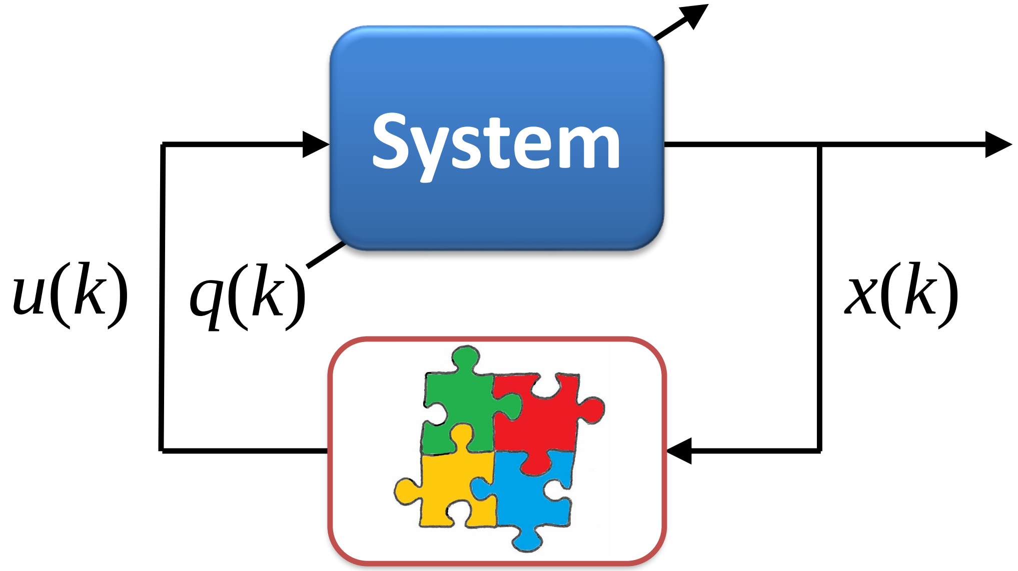
#	Σ	n_v	n_x	n_u



1.7 GHz i5 CPU with Matlab R2013a and solver MOSEK

Case Study

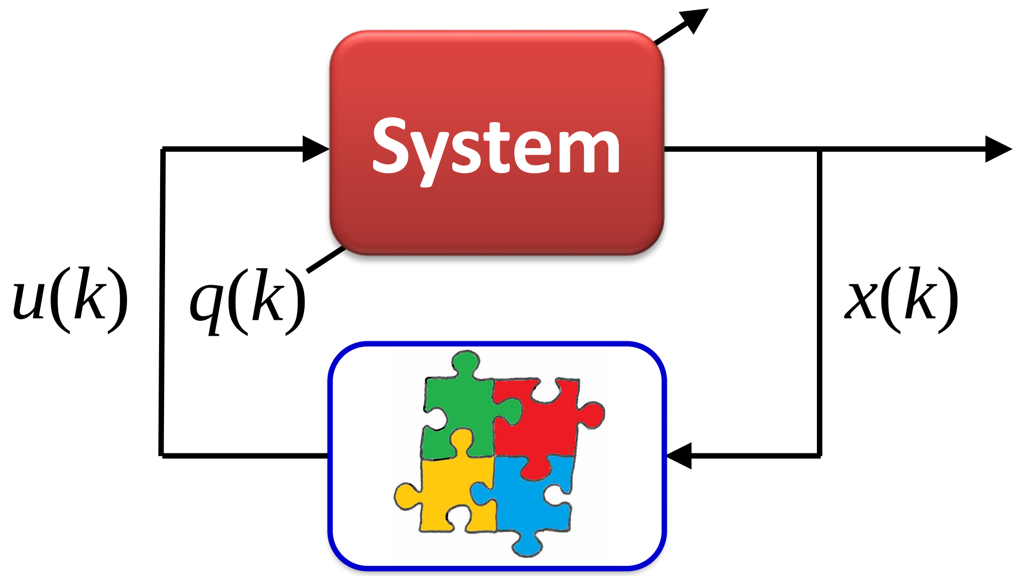
#	Σ	n_v	n_x	n_u



1.7 GHz i5 CPU with Matlab R2013a and solver MOSEK

Case Study

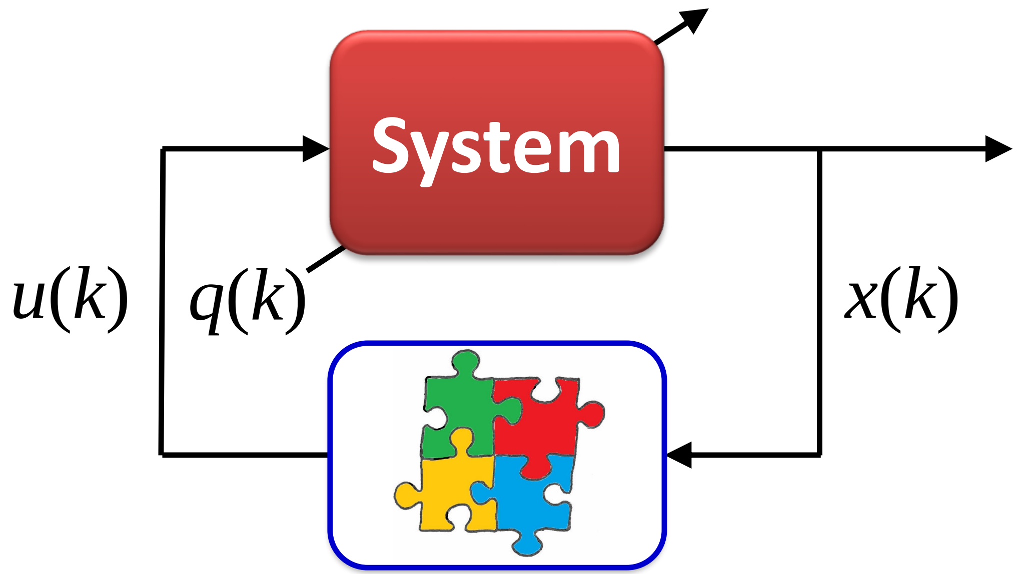
#	Σ	n_{v}	n_{x}	n_{u}
Case I	100	4	2	1



1.7 GHz i5 CPU with Matlab R2013a and solver MOSEK

Case Study

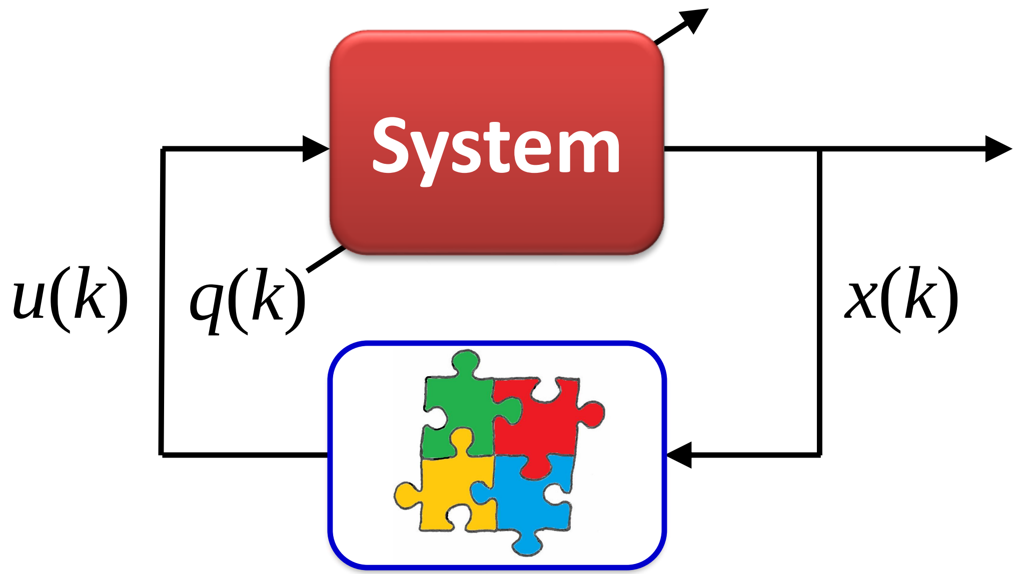
#	Σ	n_{v}	n_{x}	n_{u}
Case I	100	4	2	1
Case II	100	8	4	2



1.7 GHz i5 CPU with Matlab R2013a and solver MOSEK

Case Study

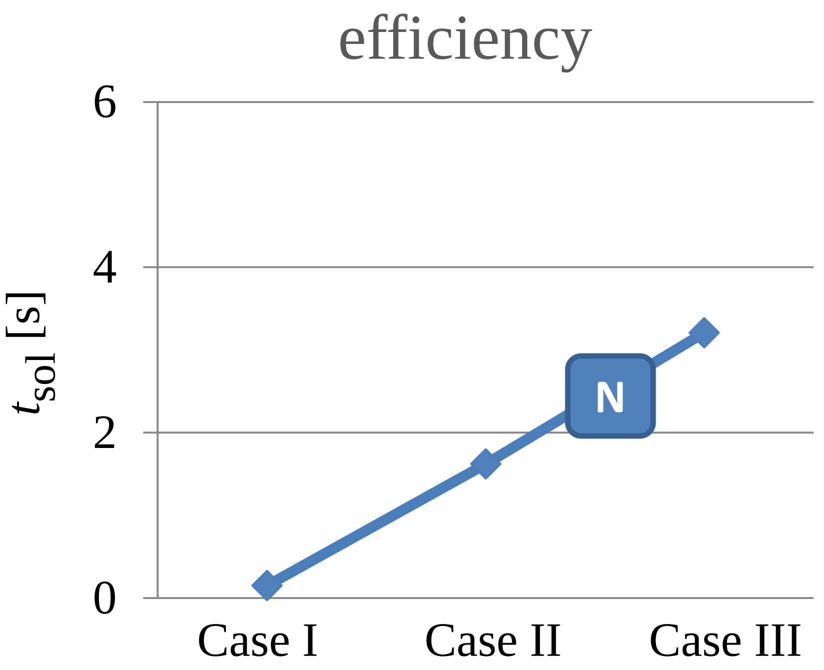
#	Σ	n_{v}	n_{x}	n_{u}
Case I	100	4	2	1
Case II	100	8	4	2
Case III	100	8	4	3



1.7 GHz i5 CPU with Matlab R2013a and solver MOSEK

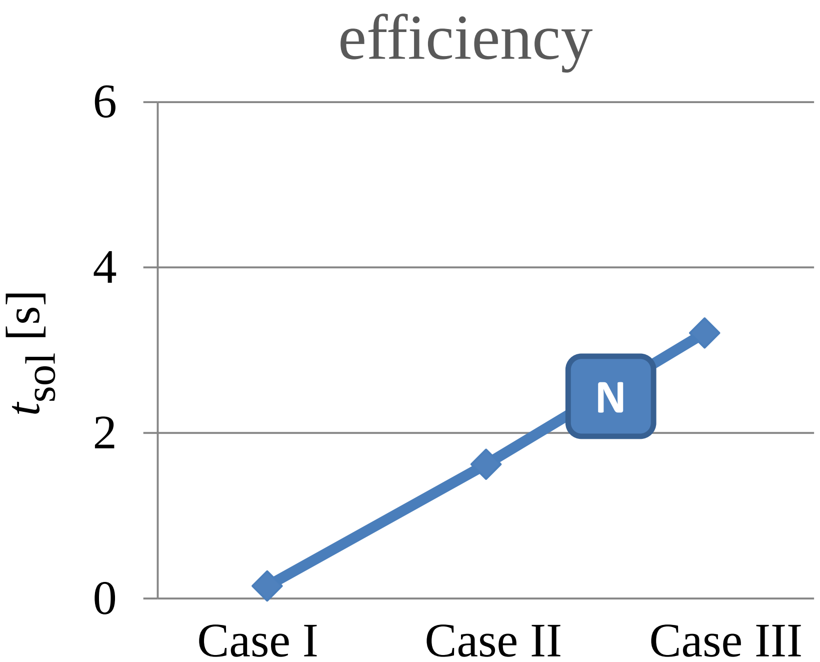
Case Study

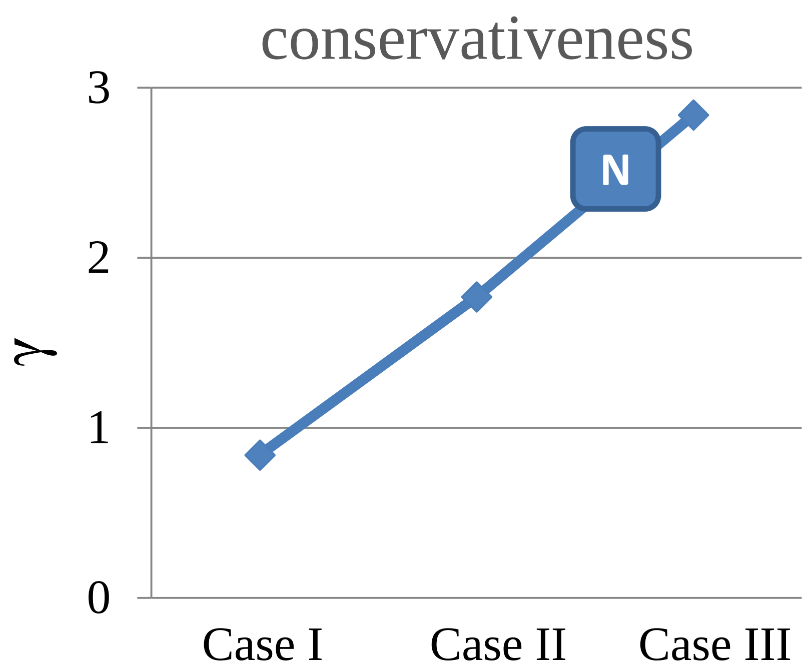
#	Σ	n_v	n_x	n_u
Case I	100	4	2	1
Case II	100	8	4	2
Case III	100	8	4	3



Case Study

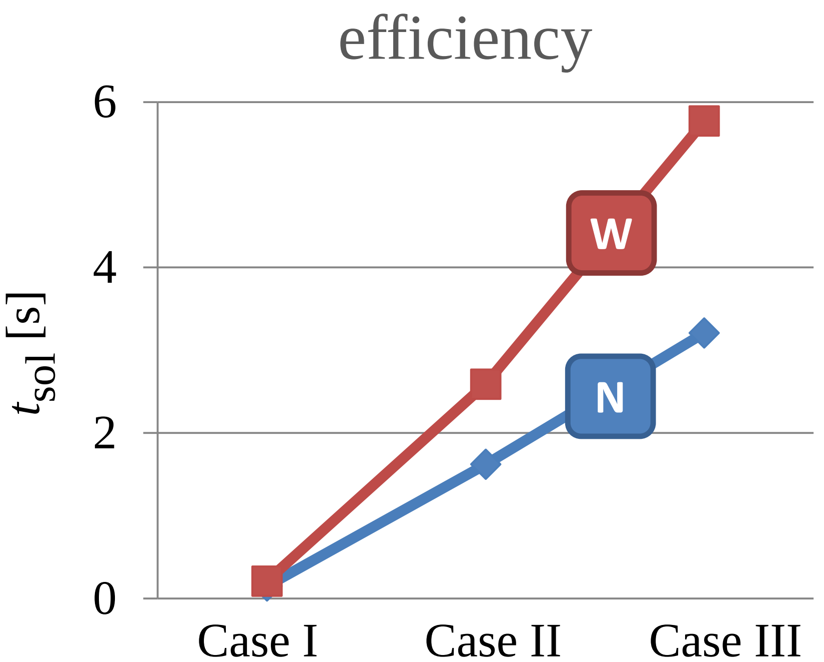
#	Σ	n_v	n_x	n_u
Case I	100	4	2	1
Case II	100	8	4	2
Case III	100	8	4	3

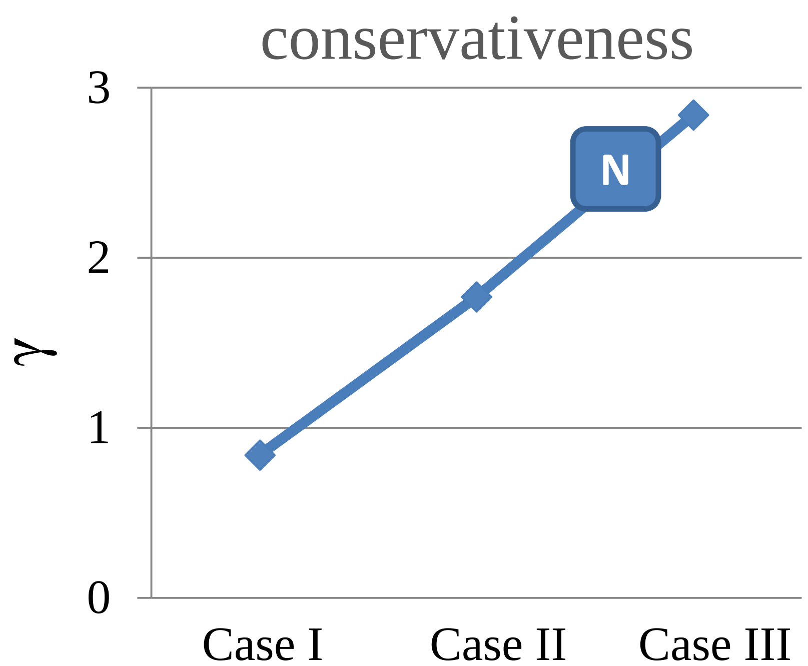




Case Study

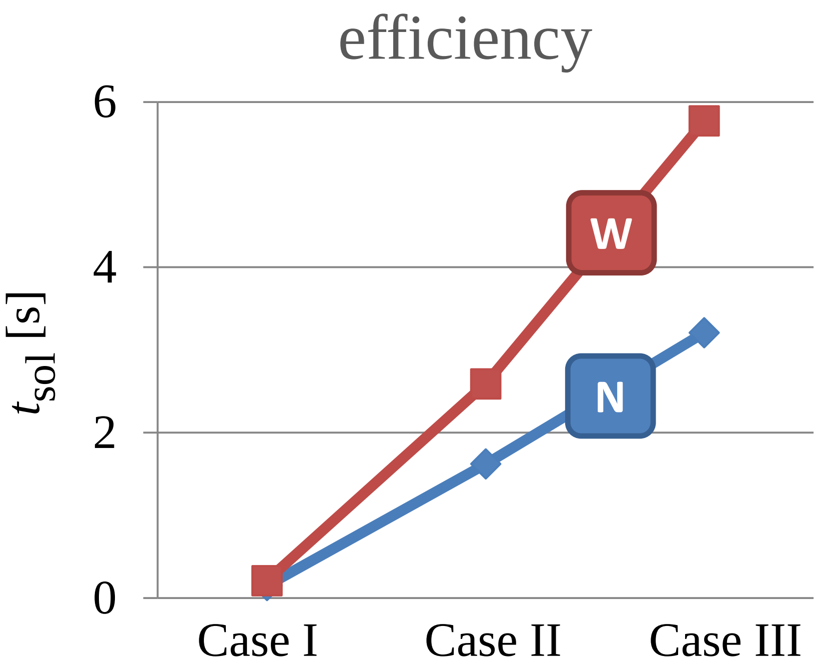
#	Σ	n_v	n_x	n_u
Case I	100	4	2	1
Case II	100	8	4	2
Case III	100	8	4	3

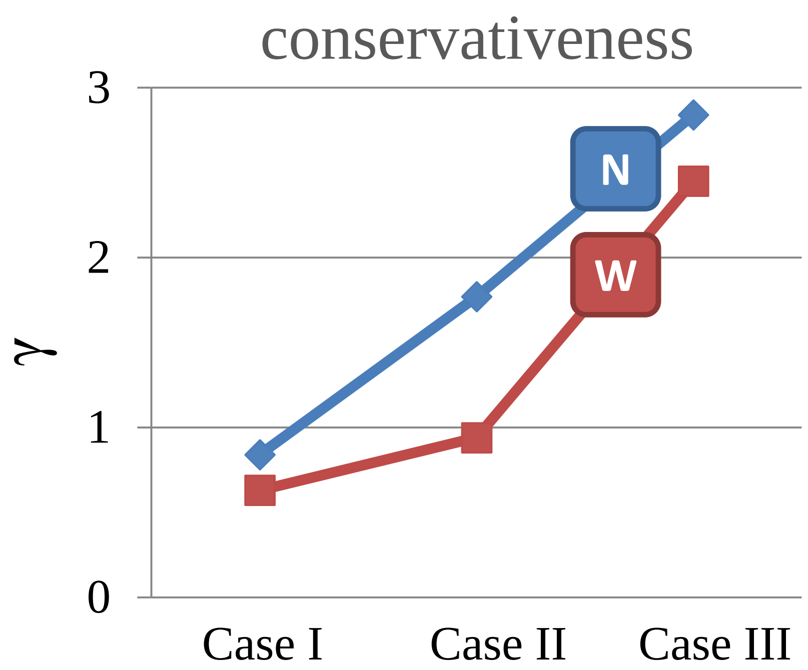




Case Study

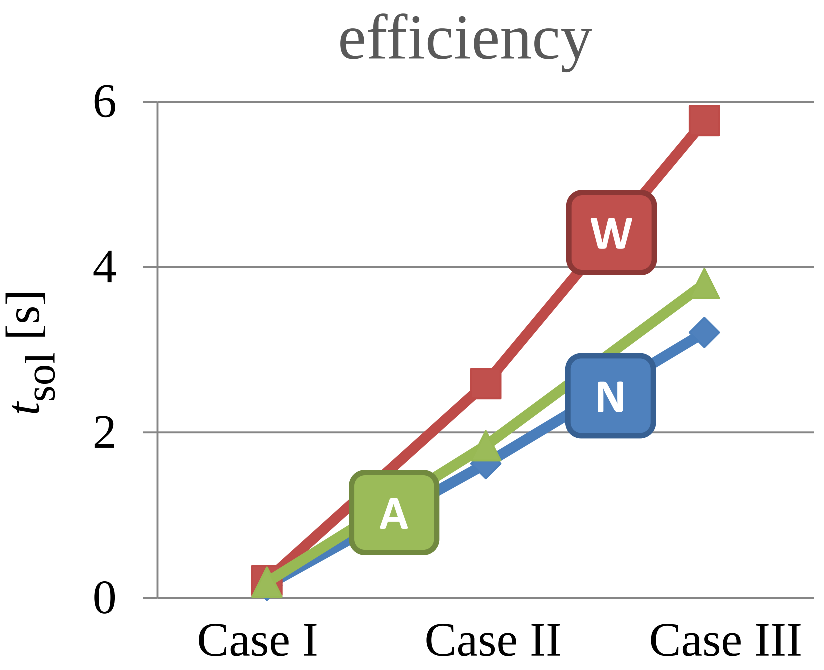
#	Σ	n_v	n_x	n_u
Case I	100	4	2	1
Case II	100	8	4	2
Case III	100	8	4	3

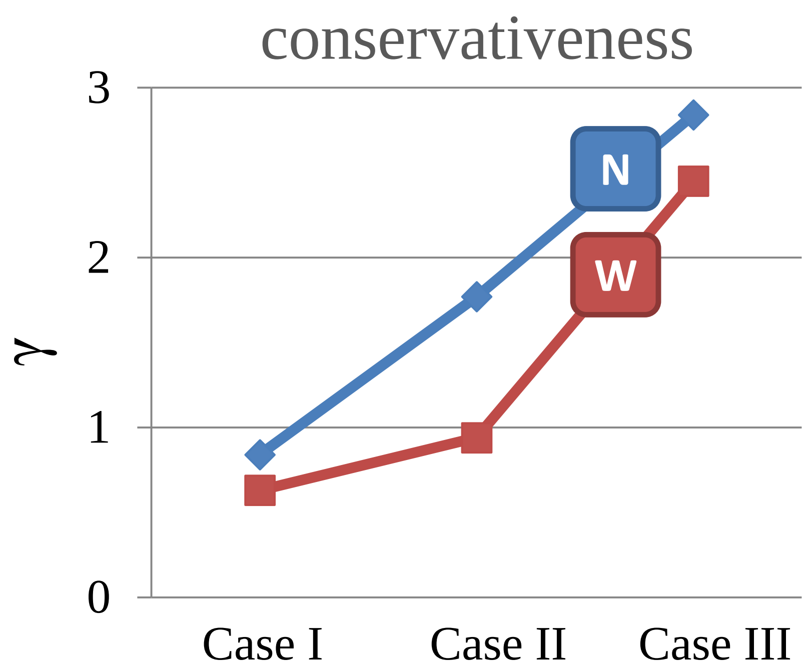




Case Study

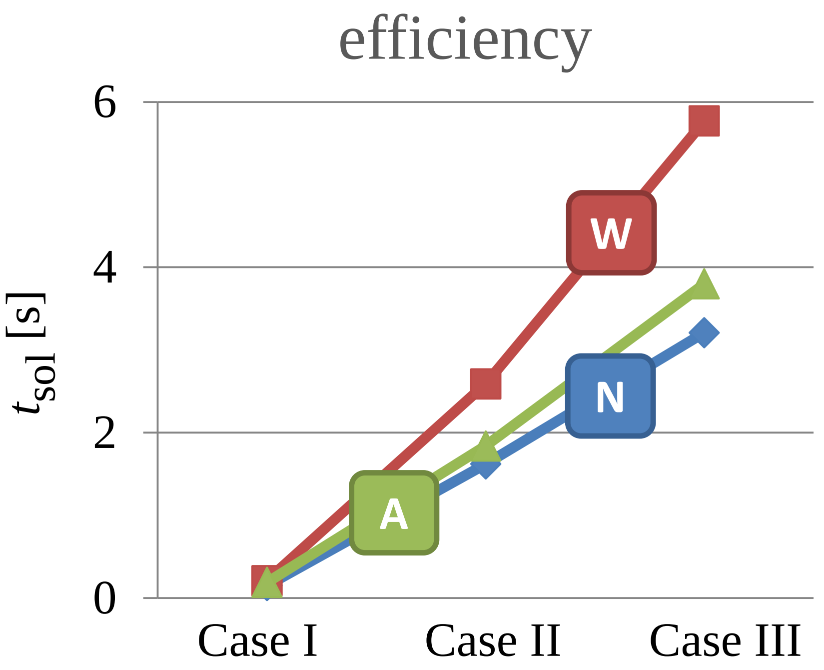
#	Σ	n_v	n_x	n_u
Case I	100	4	2	1
Case II	100	8	4	2
Case III	100	8	4	3

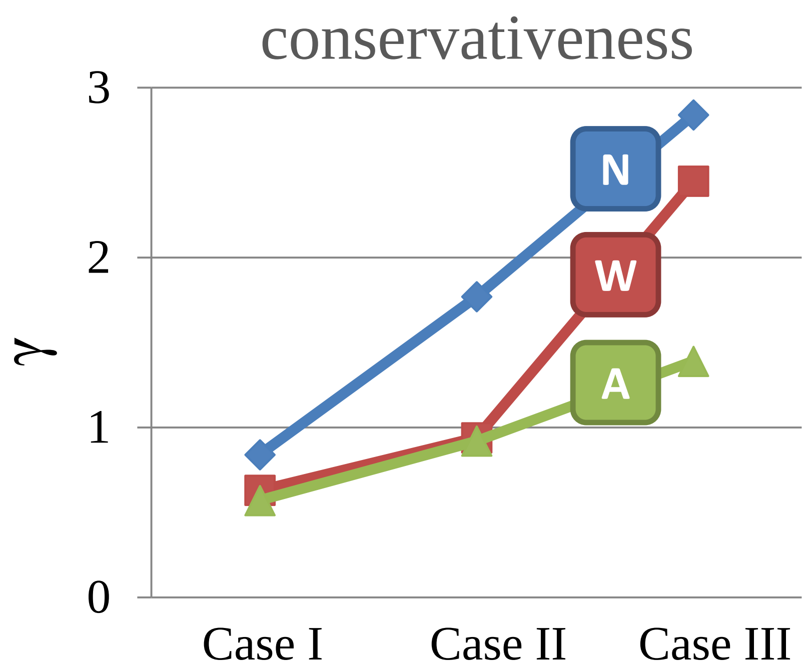




Case Study

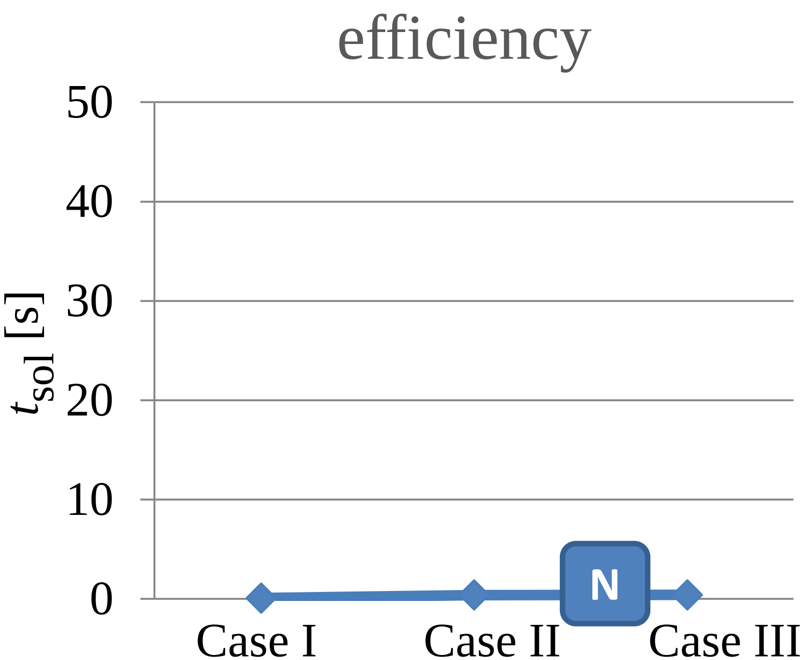
#	Σ	n_v	n_x	n_u
Case I	100	4	2	1
Case II	100	8	4	2
Case III	100	8	4	3

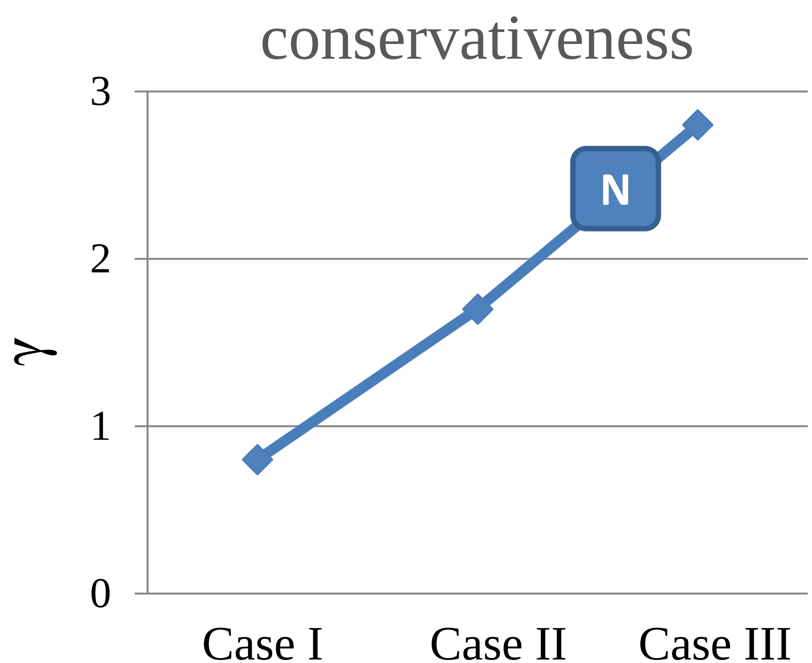




Case Study

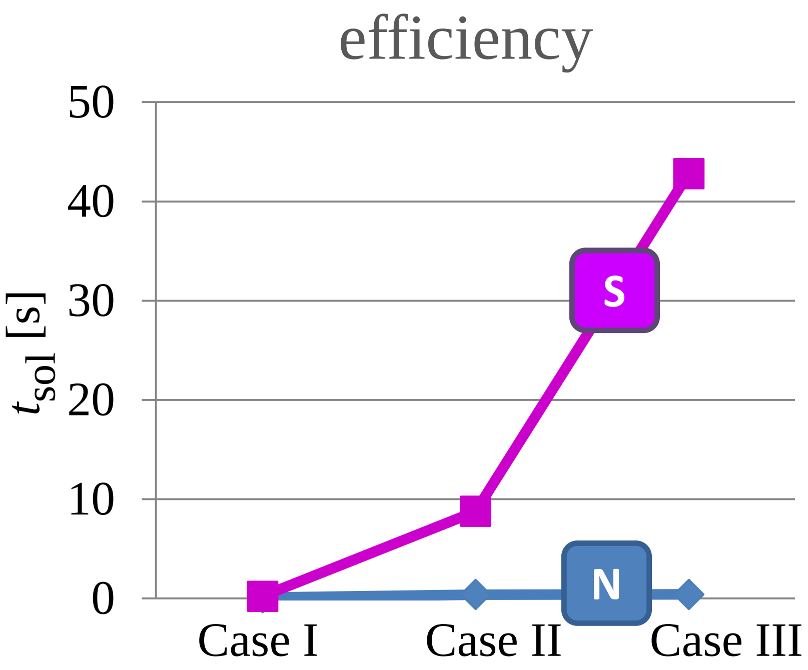
#	Σ	n_v	n_x	n_u
Case I	100	4	2	1
Case II	100	8	4	2
Case III	100	8	4	3

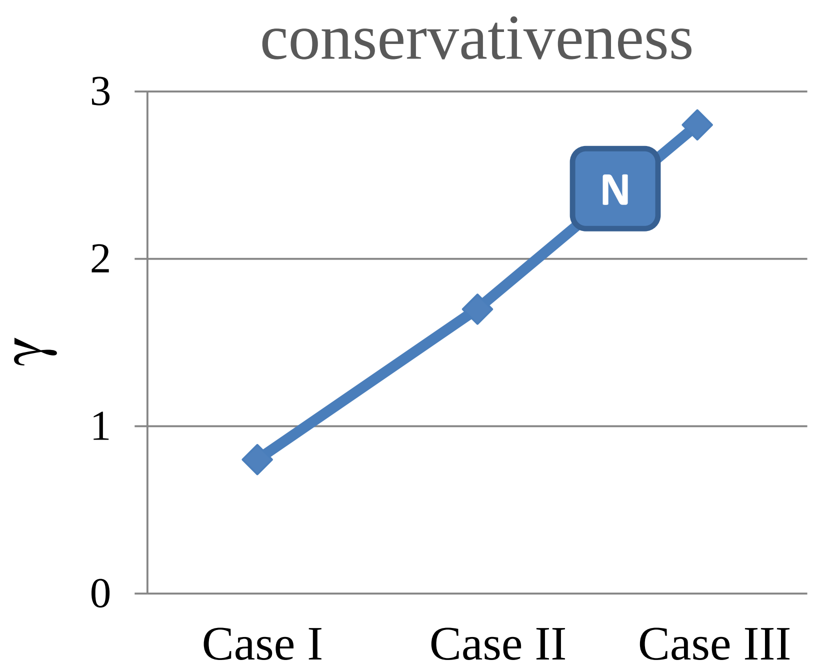




Case Study

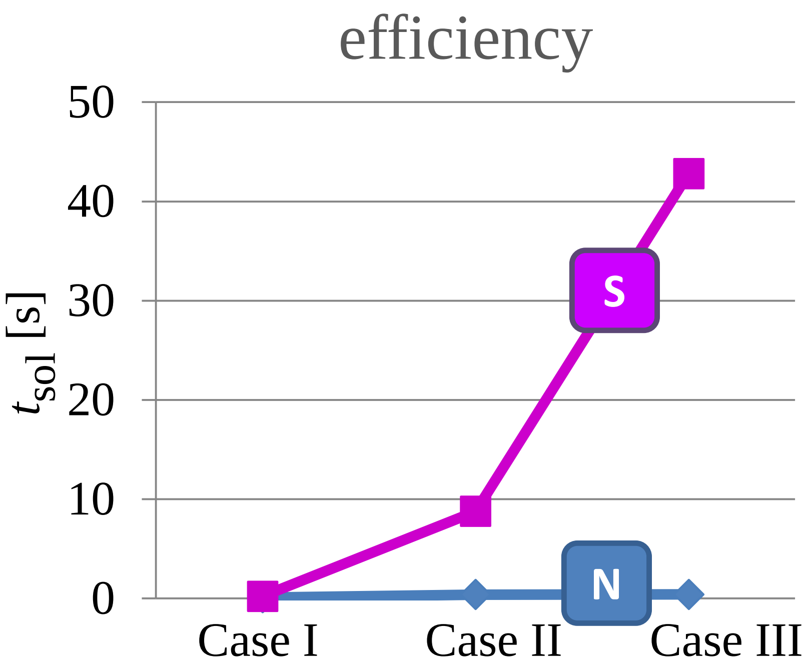
#	Σ	n_v	n_x	n_u
Case I	100	4	2	1
Case II	100	8	4	2
Case III	100	8	4	3

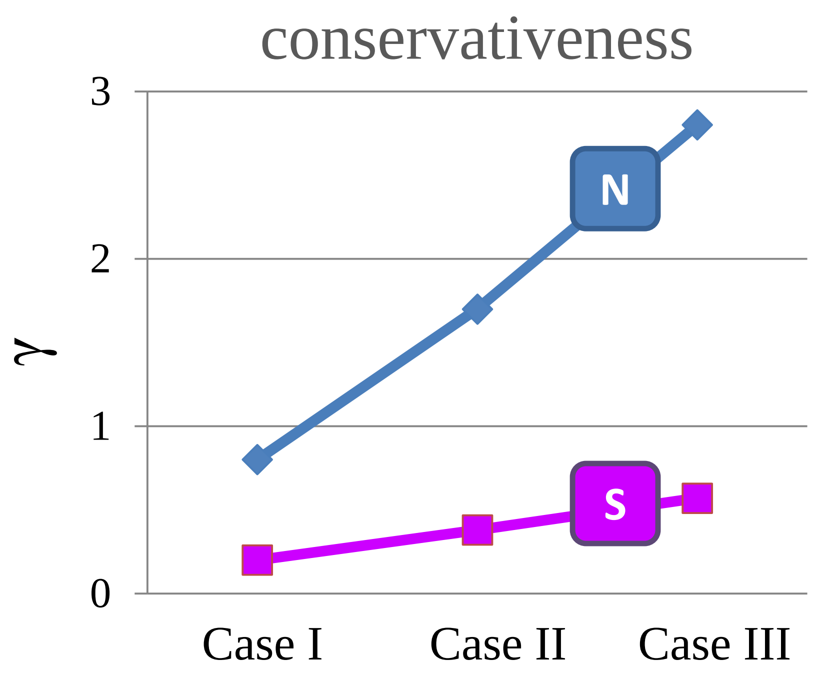




Case Study

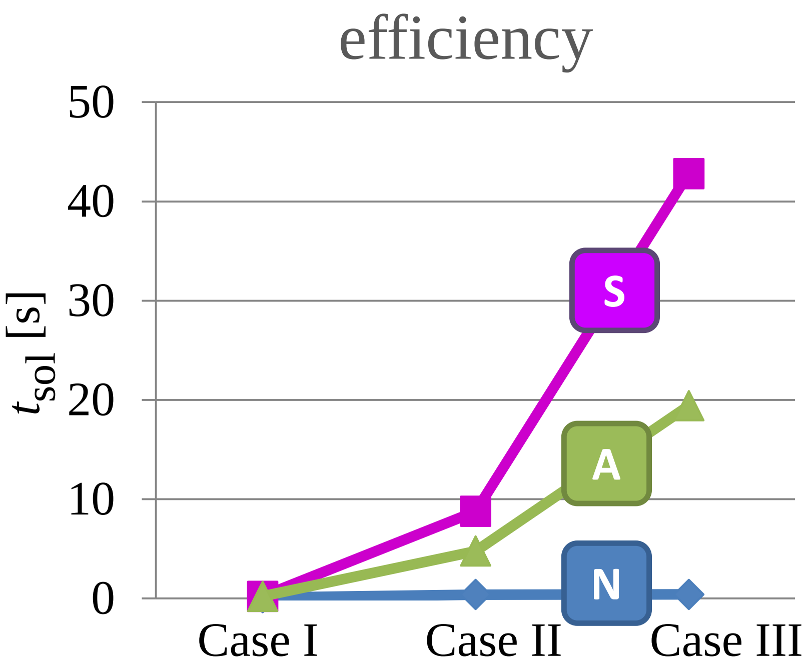
#	Σ	n_v	n_x	n_u
Case I	100	4	2	1
Case II	100	8	4	2
Case III	100	8	4	3

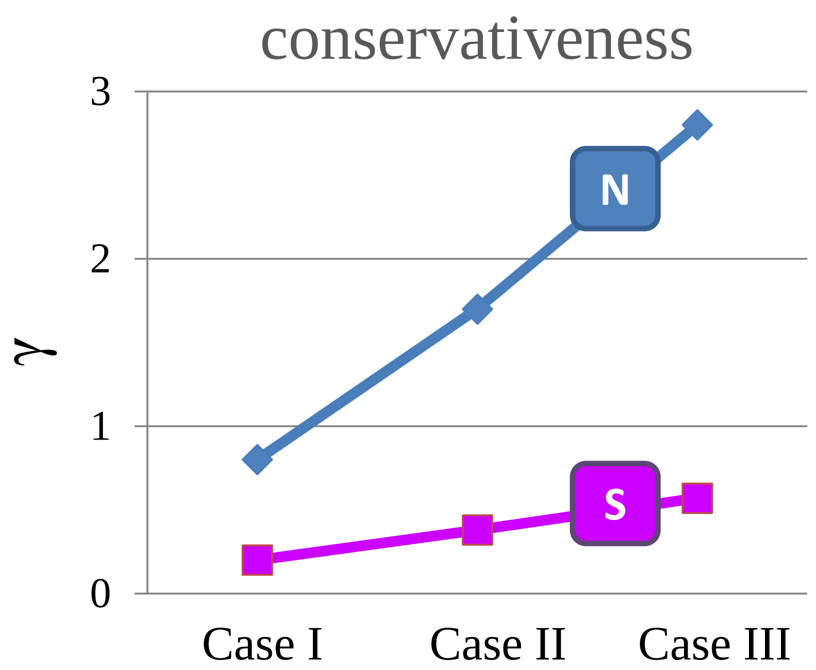




Case Study

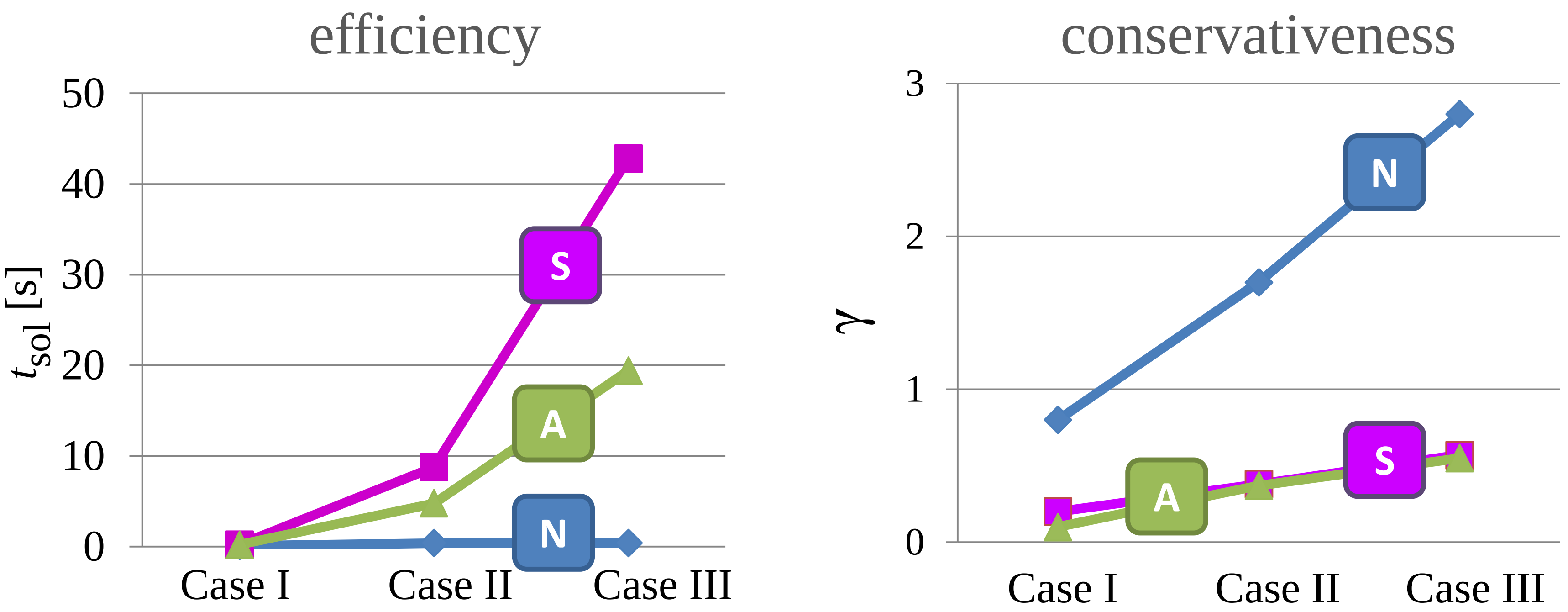
#	Σ	n_v	n_x	n_u
Case I	100	4	2	1
Case II	100	8	4	2
Case III	100	8	4	3



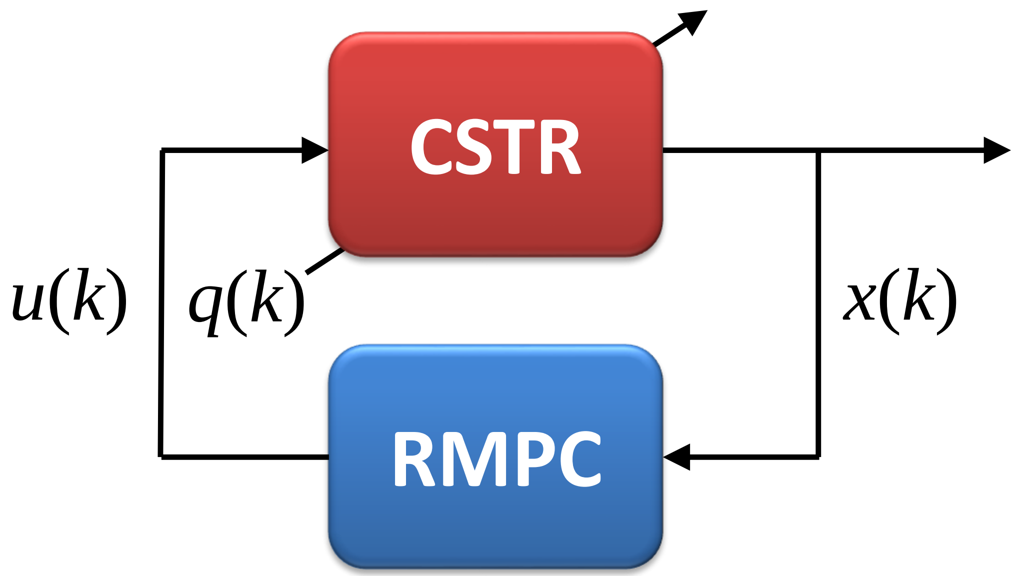


Case Study

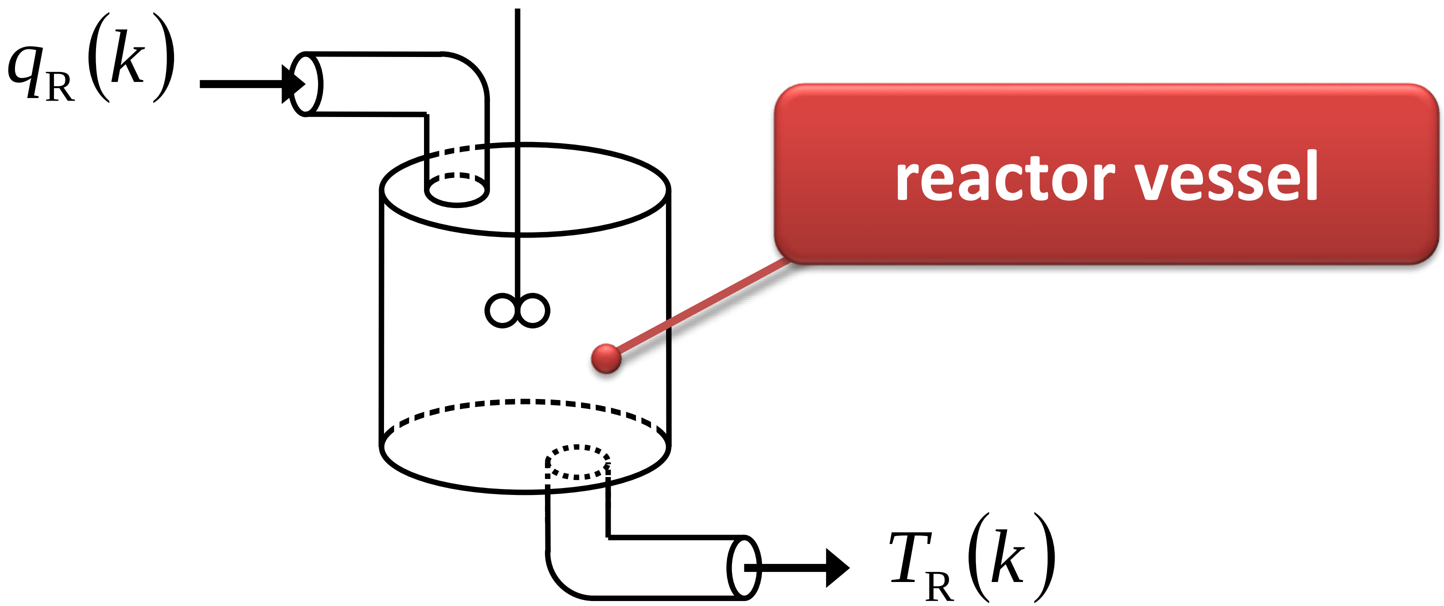
#	Σ	n_v	n_x	n_u
Case I	100	4	2	1
Case II	100	8	4	2
Case III	100	8	4	3



Control of CSTR

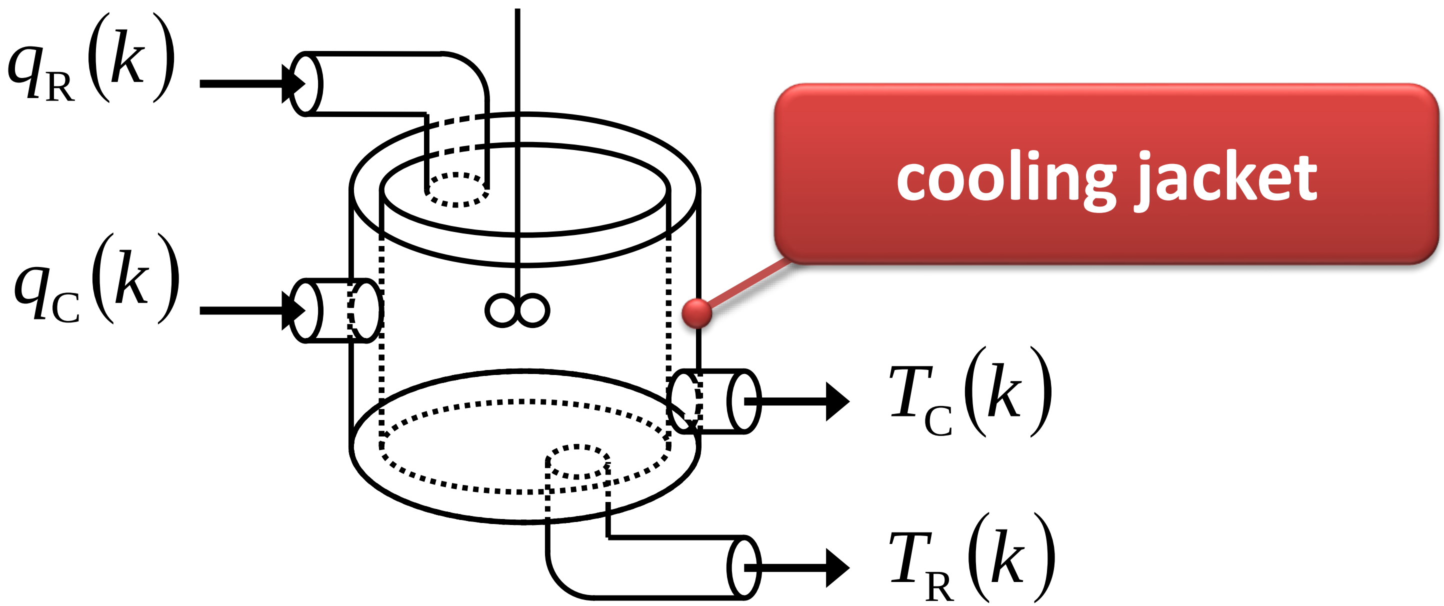


$$x(k+1) = A\,x(k) + B\,u(k), \qquad x(0) = x_0,$$
$$y(k) = C\,x(k), \qquad y \in \mathbb{Y},$$
$$[A, B] \in \mathrm{convhull}\{[A_v, B_v]\}, \qquad u \in \mathbb{U}.$$



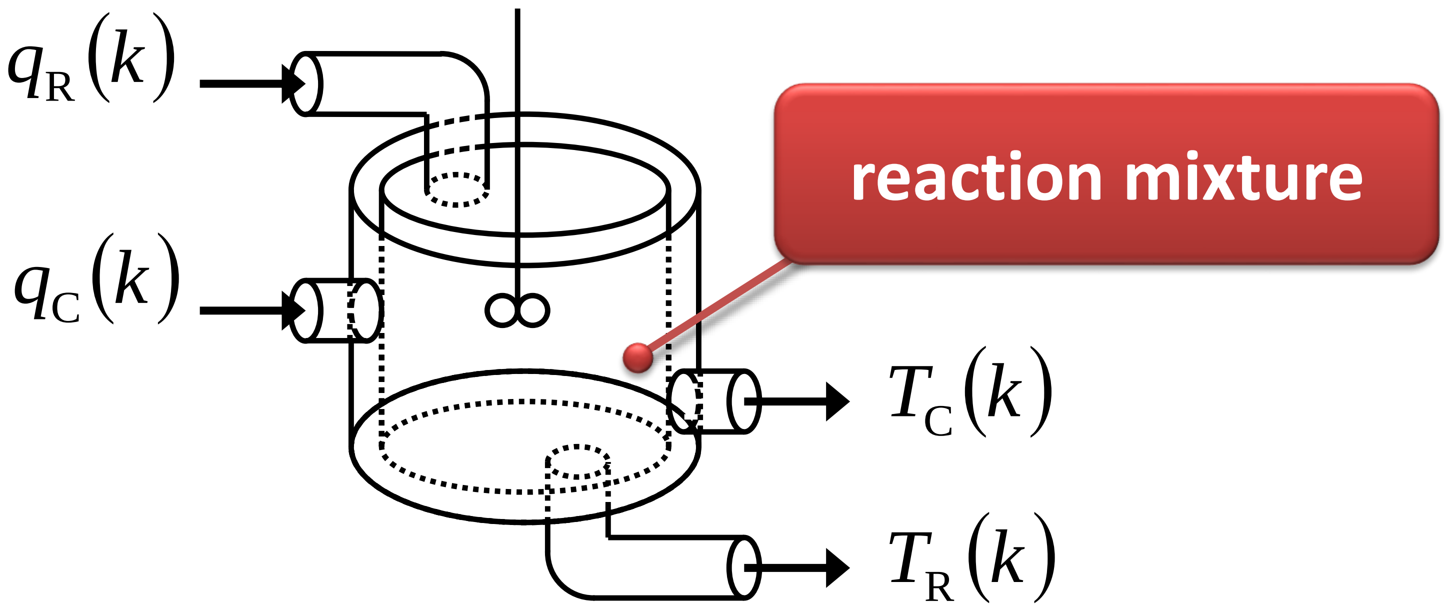
parameter [unit]	reactor vessel	
V [m^3]	2.400	
ρ [kg m^{-3}]	947.2	
c_P [kJ kg^{-1} K^{-1}]	3.719	
q^S [m^3 min^{-1}]	0.072	
T_{in} [K]	299.1	

Molnár et al., Chemical Papers , 2002.



parameter [unit]	reactor vessel	cooling jacket
V [m^3]	2.400	2.000
ρ [$kg\ m^{-3}$]	947.2	998.0
c_P [$kJ\ kg^{-1}\ K^{-1}$]	3.719	4.182
q^S [$m^3\ min^{-1}$]	0.072	0.631
T_{in} [K]	299.1	288.6

Molnár et al., Chemical Papers , 2002.



parameter [unit]	reactor vessel	cooling jacket
V [m^3]	2.400	2.000
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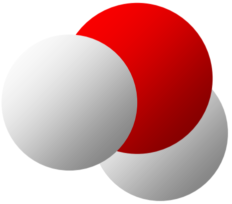
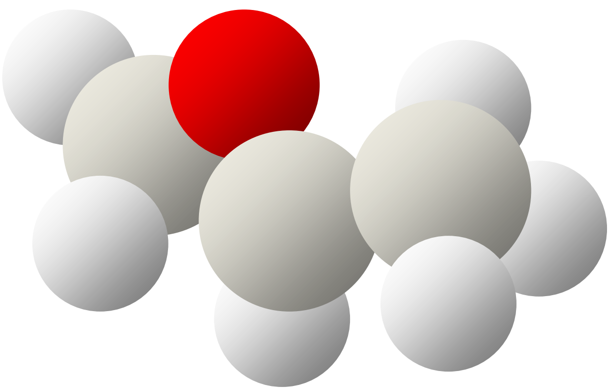
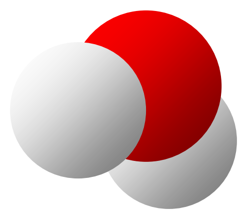
Molnár et al., Chemical Papers , 2002.



water

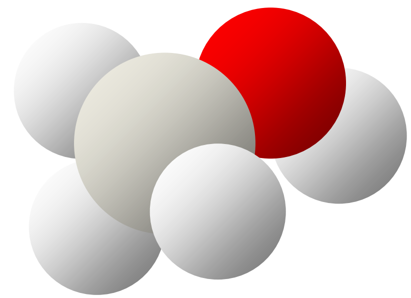


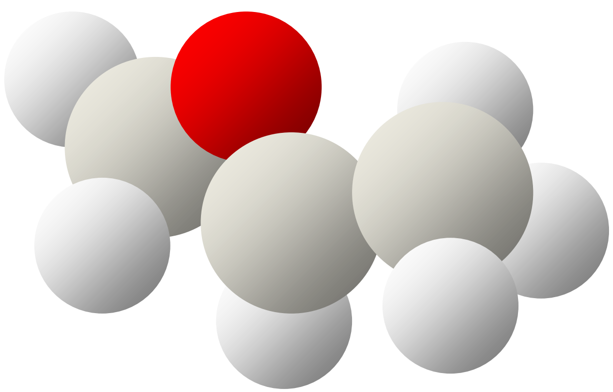
propylene oxide



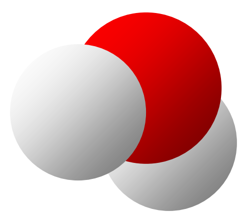
water

methanol

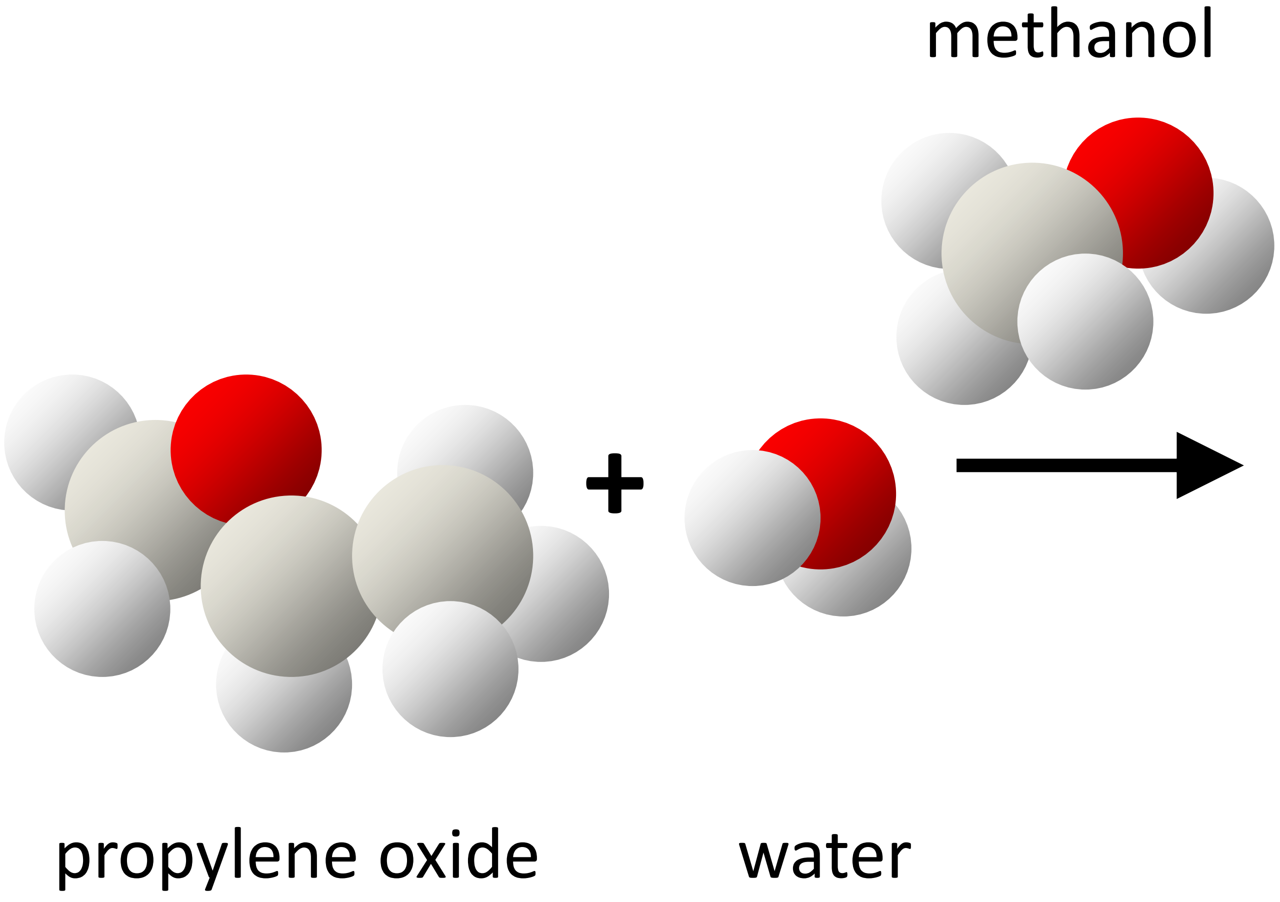


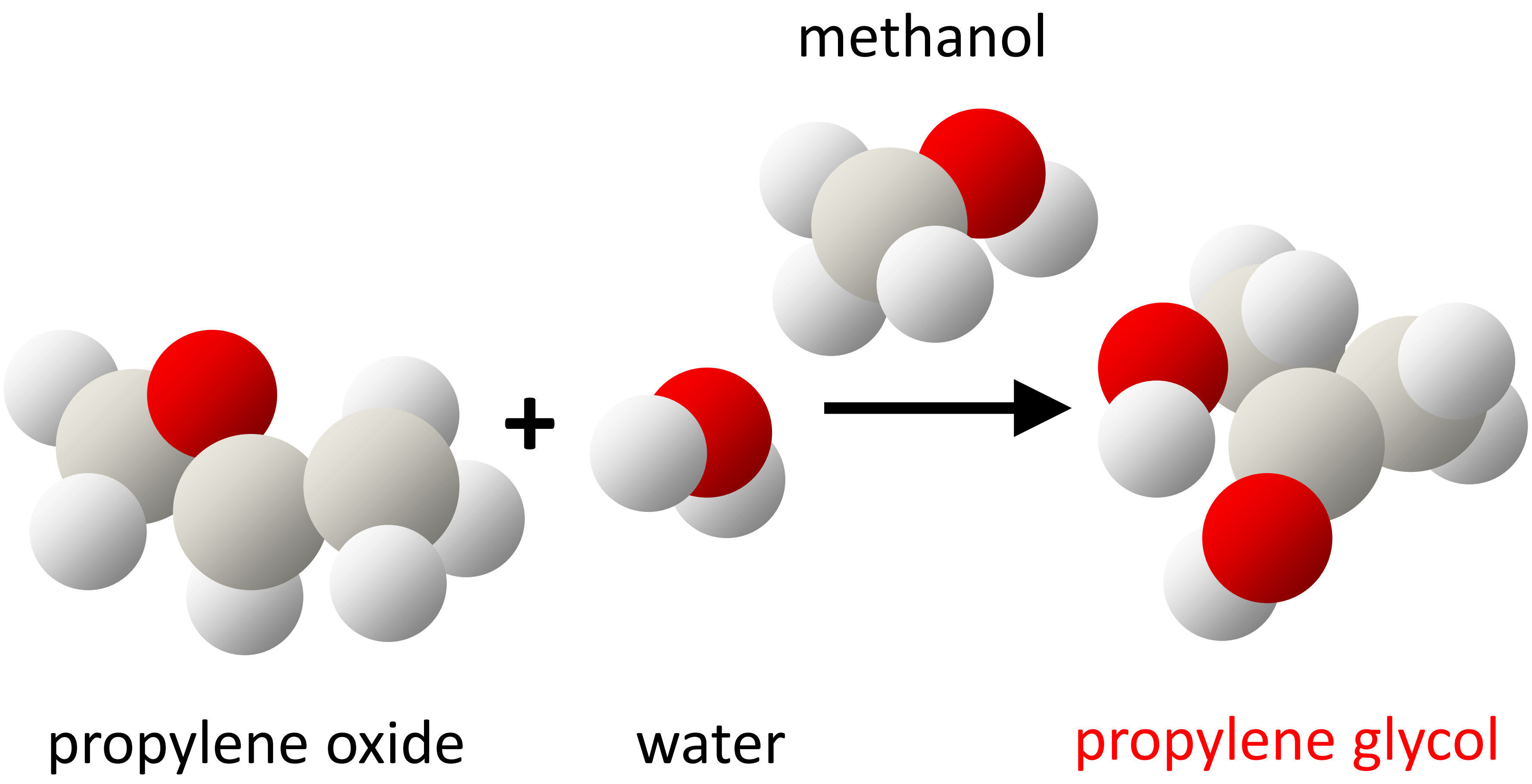


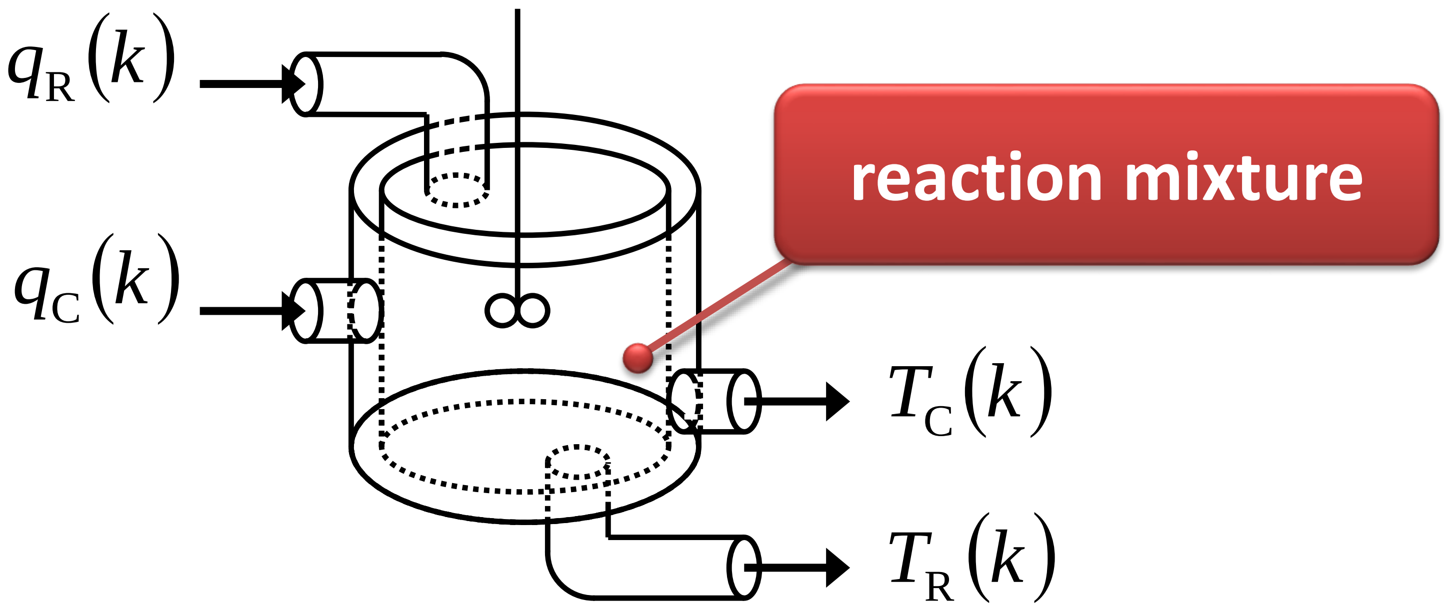
propylene oxide



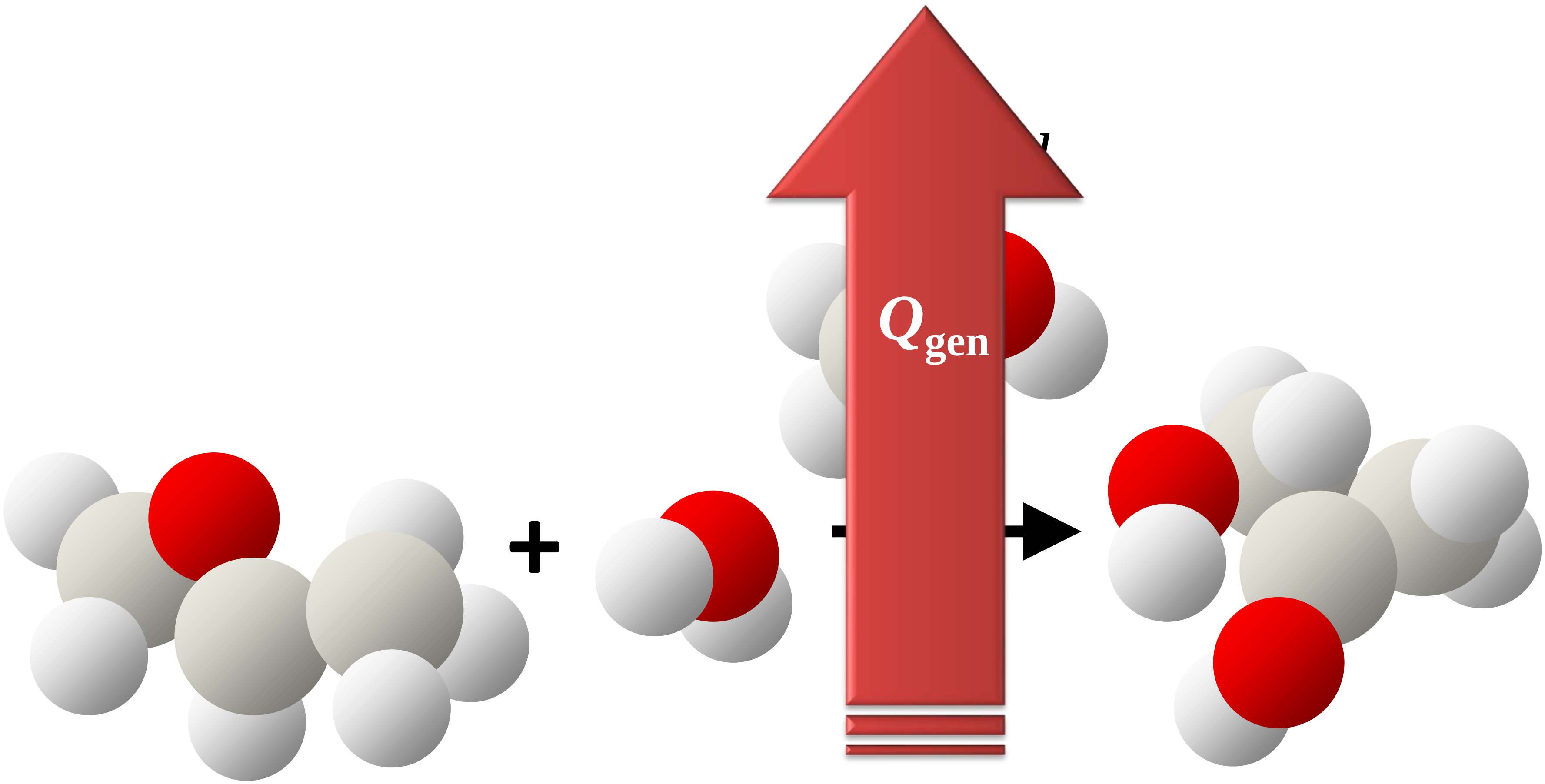
water


methanol
+
propylene oxide
water

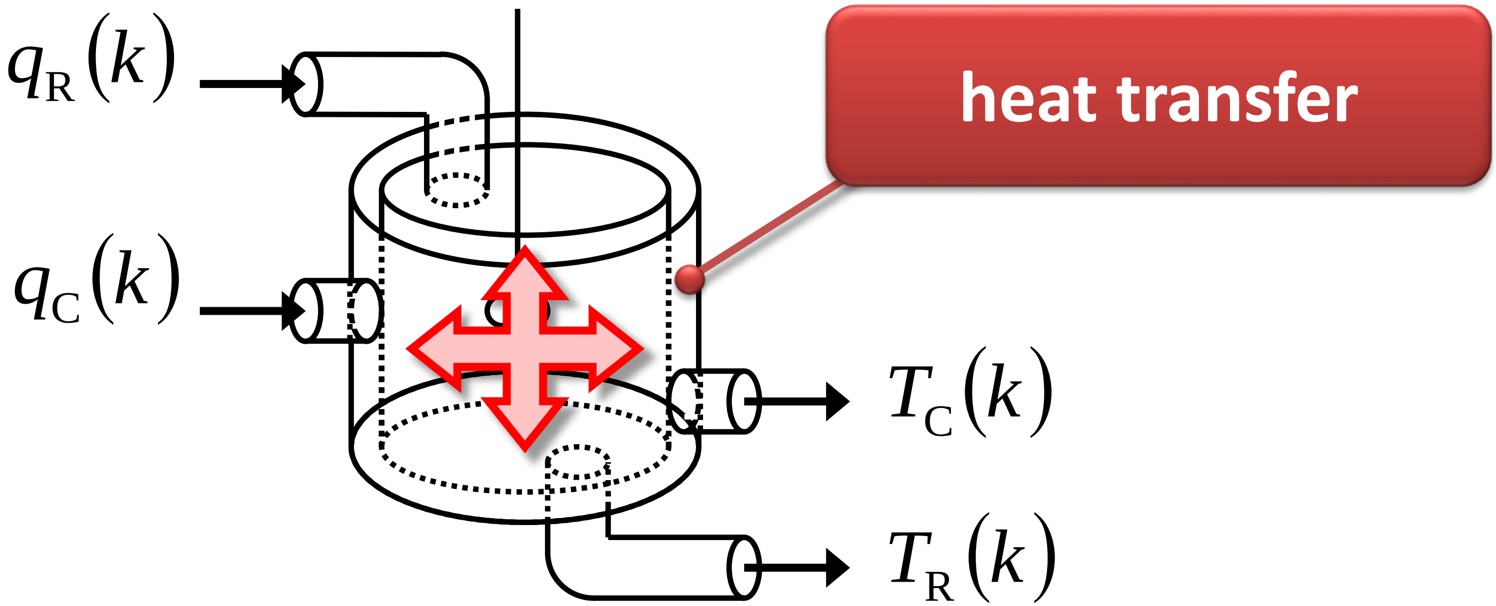

methanol
propylene oxide
water
propylene glycol



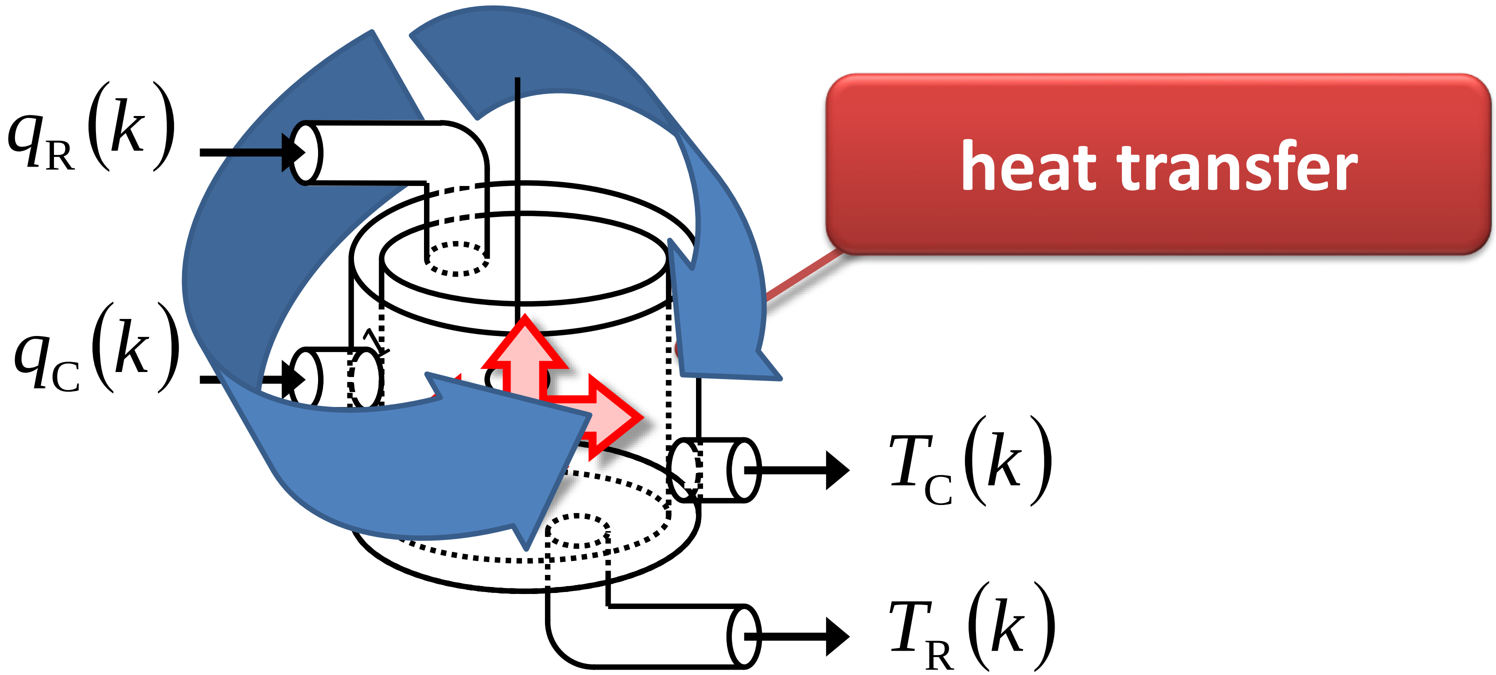
parameter [unit]	reaction mixture
$c_{PO,in}$ [kmol m^{-3}]	82.4×10^{-3}
$c_{PG,in}$ [kmol m^{-3}]	0.0×10^{-3}
E_a/R [K]	10183.0
ΔH_r [kJ kmol^{3}]	$(-5.64, -5.28)\times10^{6}$
k_∞ [min^{-1}]	$(2.407, 3.247)\times10^{11}$



propylene oxide *water* *propylene glycol*

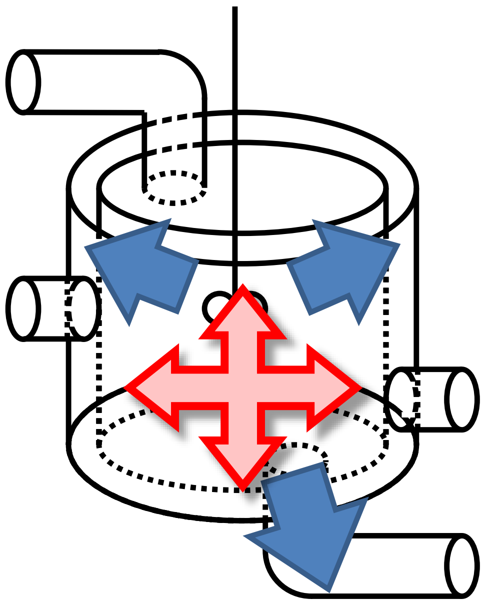


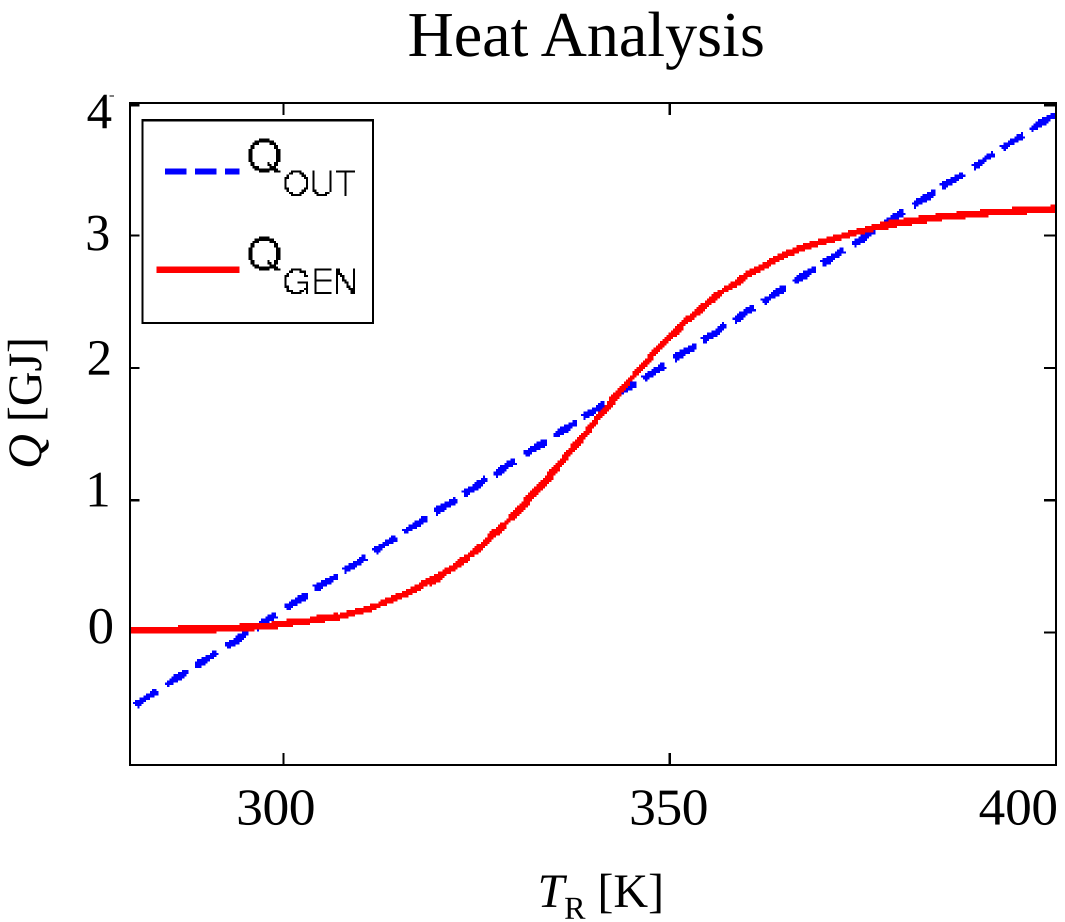
$q_{\mathrm{R}}(k)$
$q_{\mathrm{C}}(k)$
heat transfer
$T_{\mathrm{C}}(k)$
$T_{\mathrm{R}}(k)$



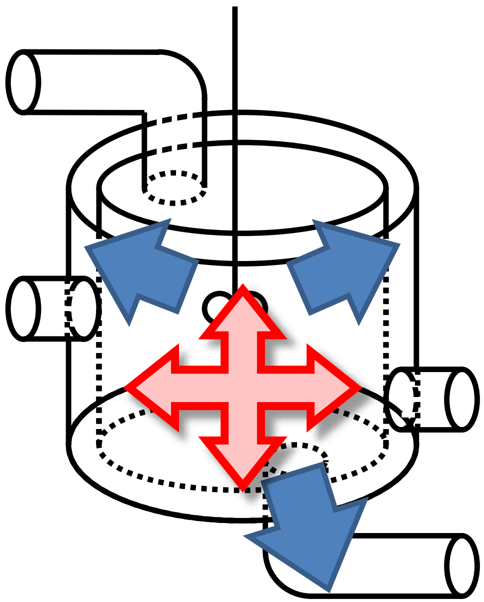
parameter [unit]	heat transfer
A_H [m^2]	8.695
U [kJ min^{-1} m^{-2} K^{-1}]	$(-5.64, -5.28) \times 10^6$

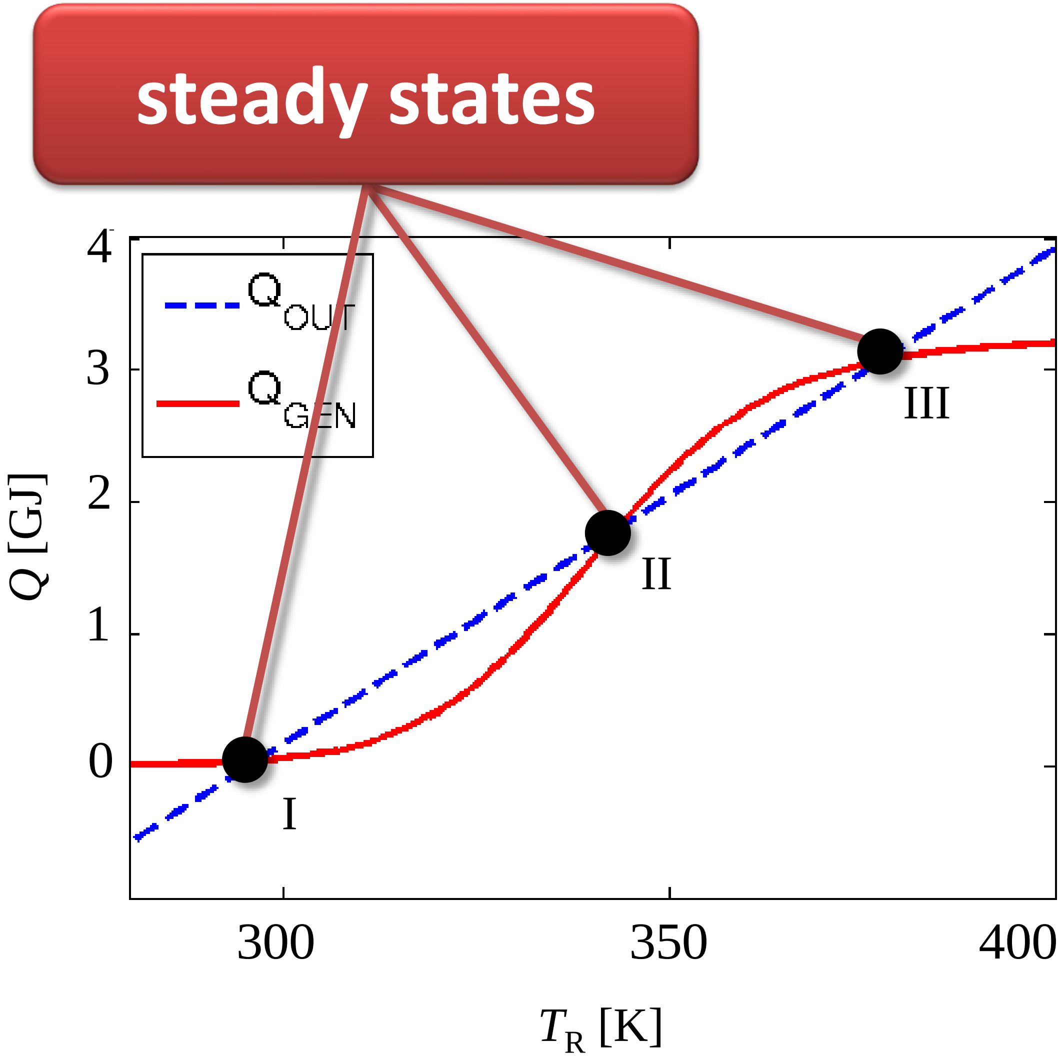
Stability Analysis



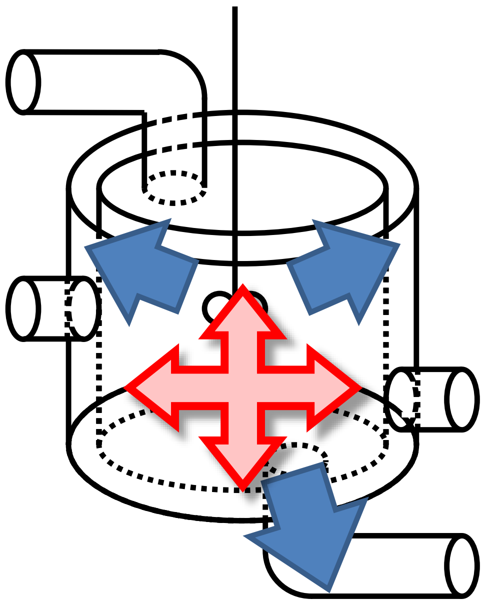


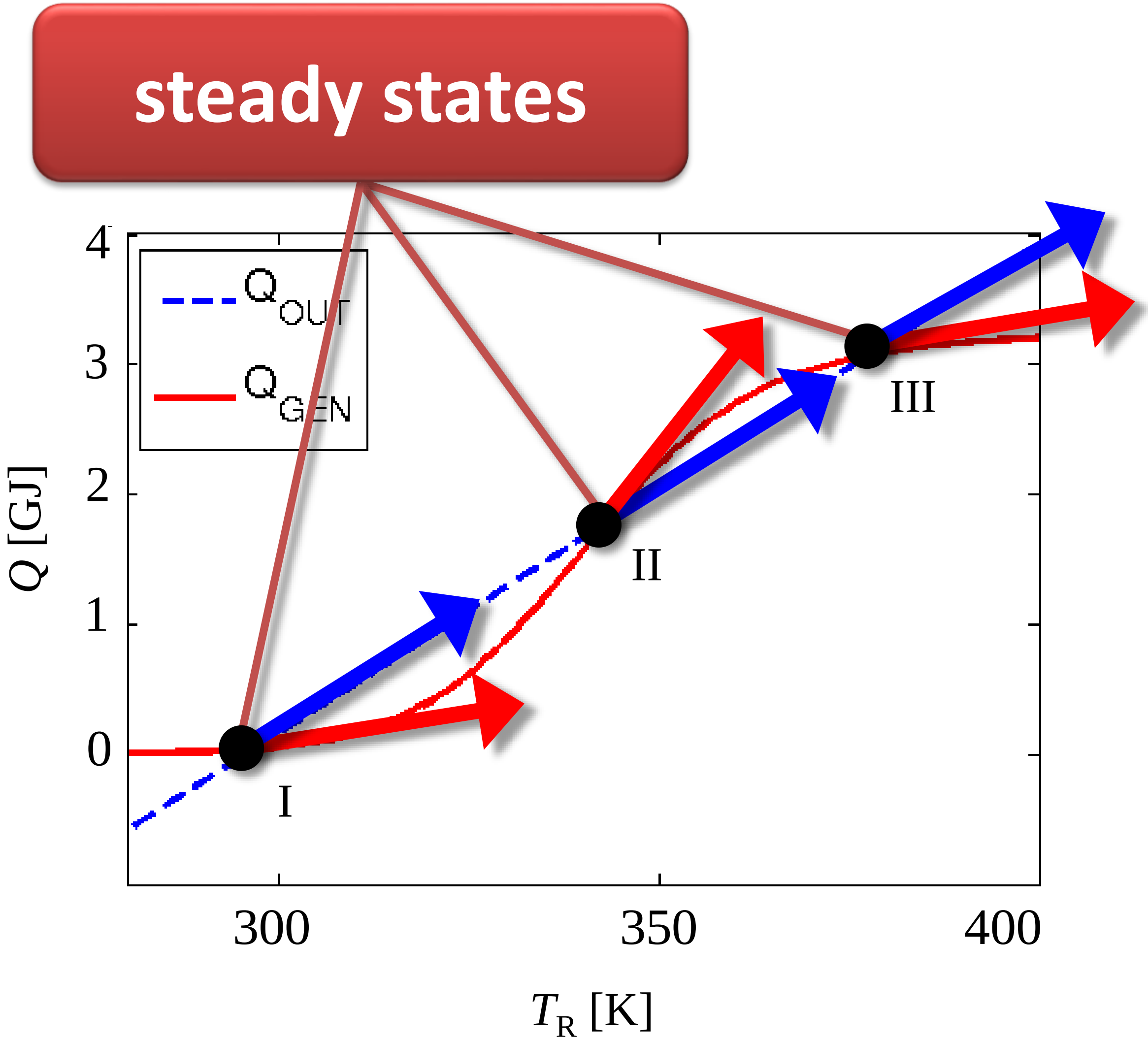
Stability Analysis





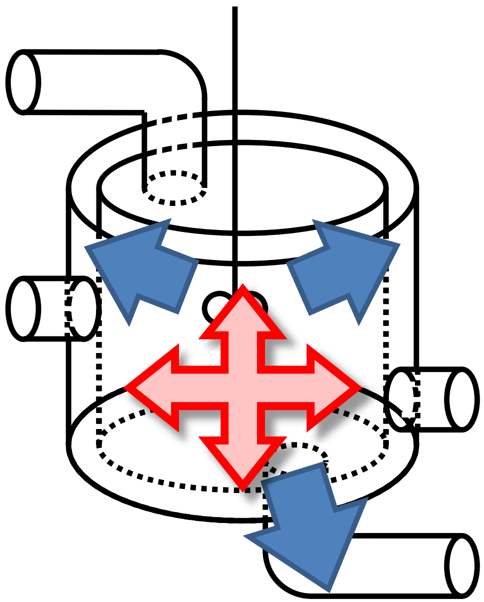
Stability Analysis

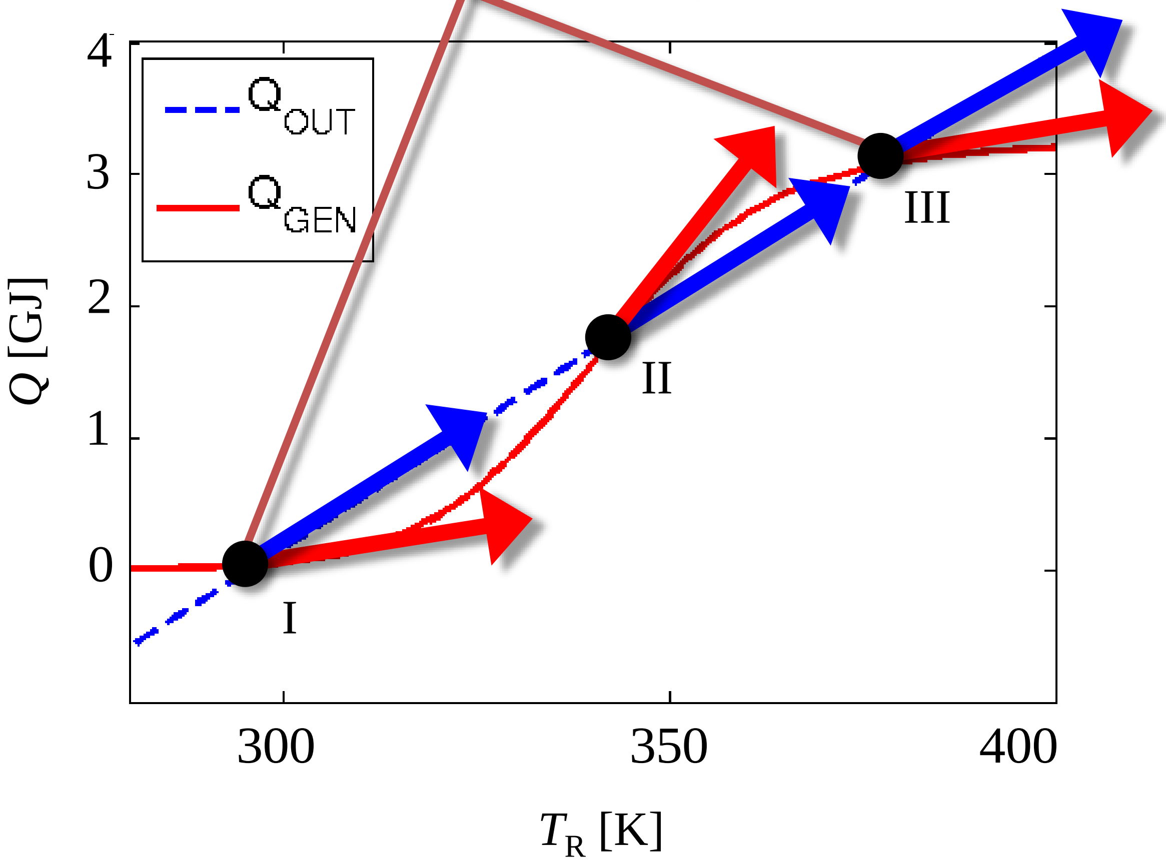




Stability Analysis

$$\frac{\partial Q_{\mathrm{OUT}}}{\partial T_{\mathrm{R}}} > \frac{\partial Q_{\mathrm{GEN}}}{\partial T_{\mathrm{R}}} \Rightarrow$$ **stable steady states**

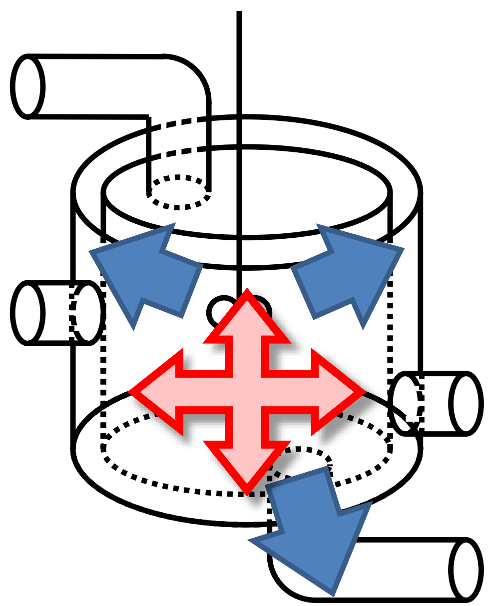


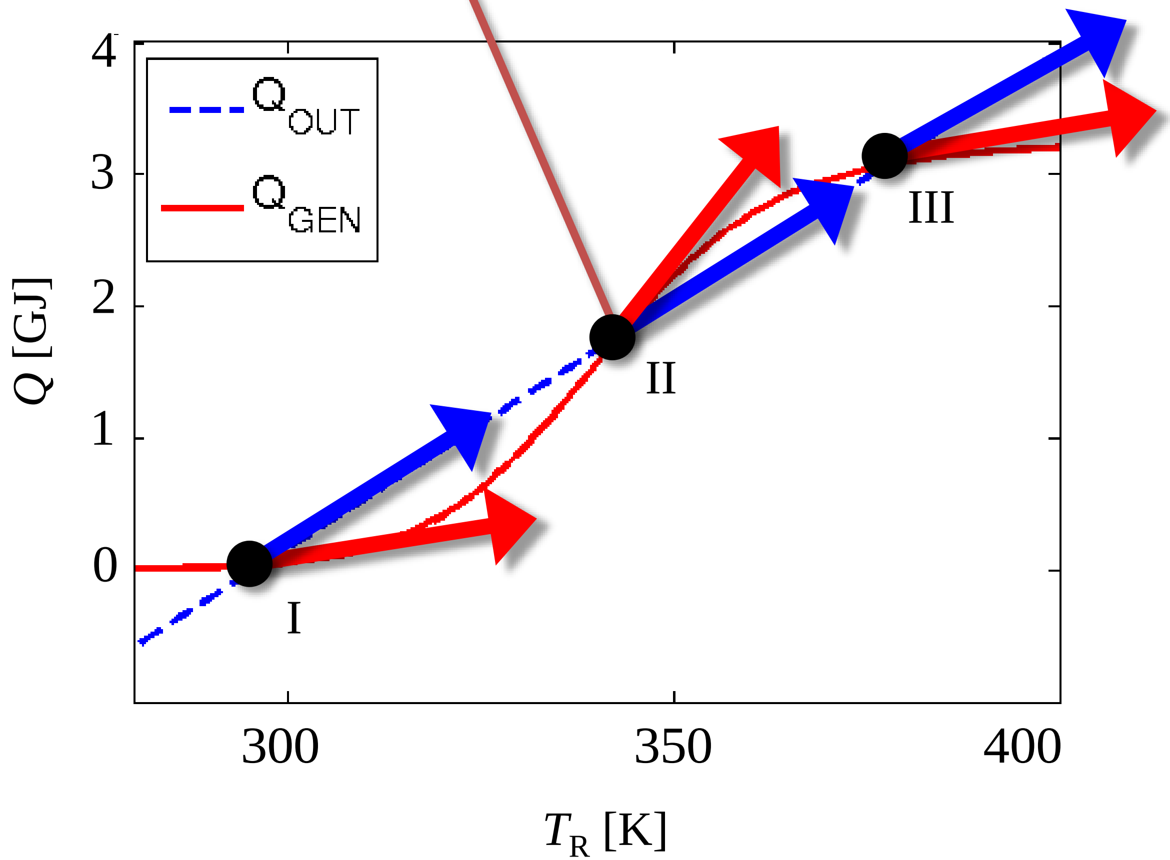


Stability Analysis

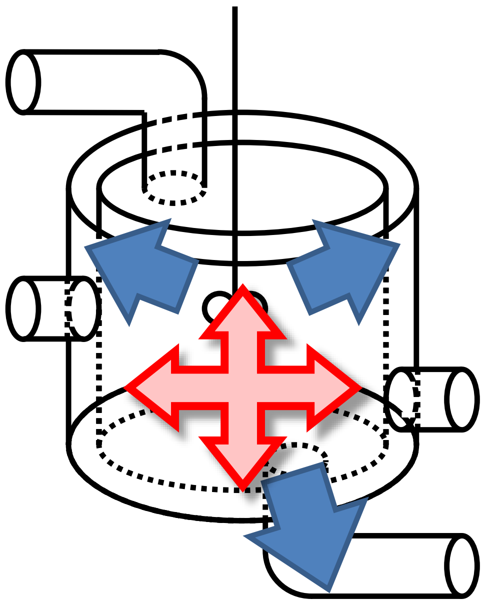
$$\frac{\partial Q_{\mathrm{OUT}}}{\partial T_{\mathrm{R}}} \not> \frac{\partial Q_{\mathrm{GEN}}}{\partial T_{\mathrm{R}}} \Rightarrow$$

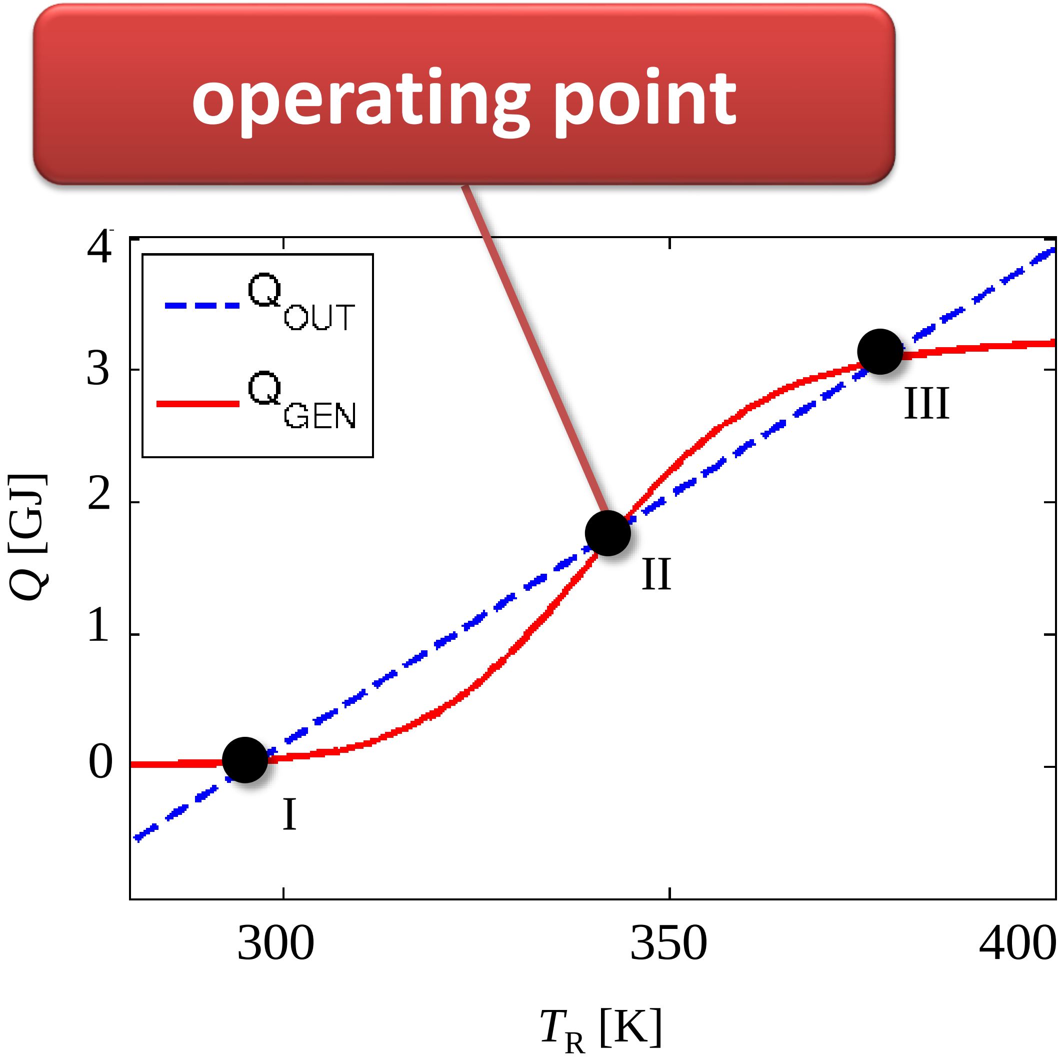
unstable steady state



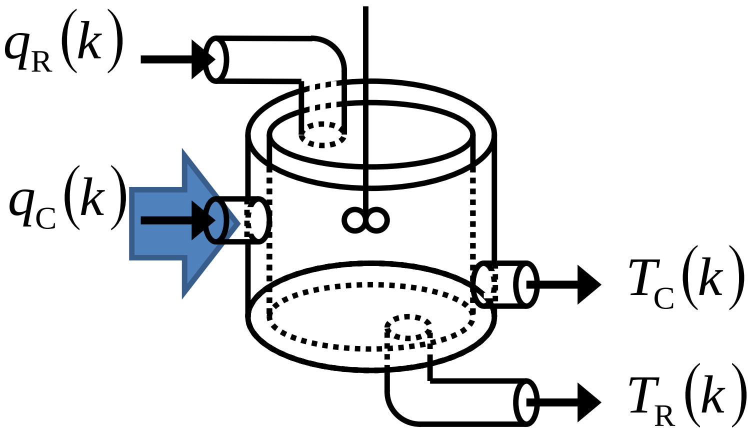


Stability Analysis

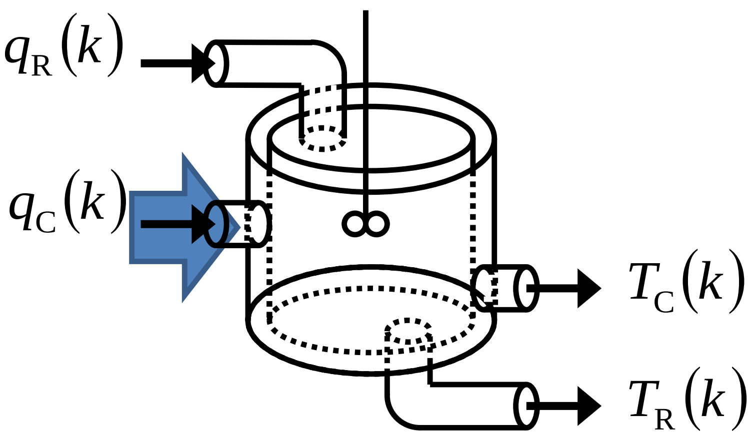


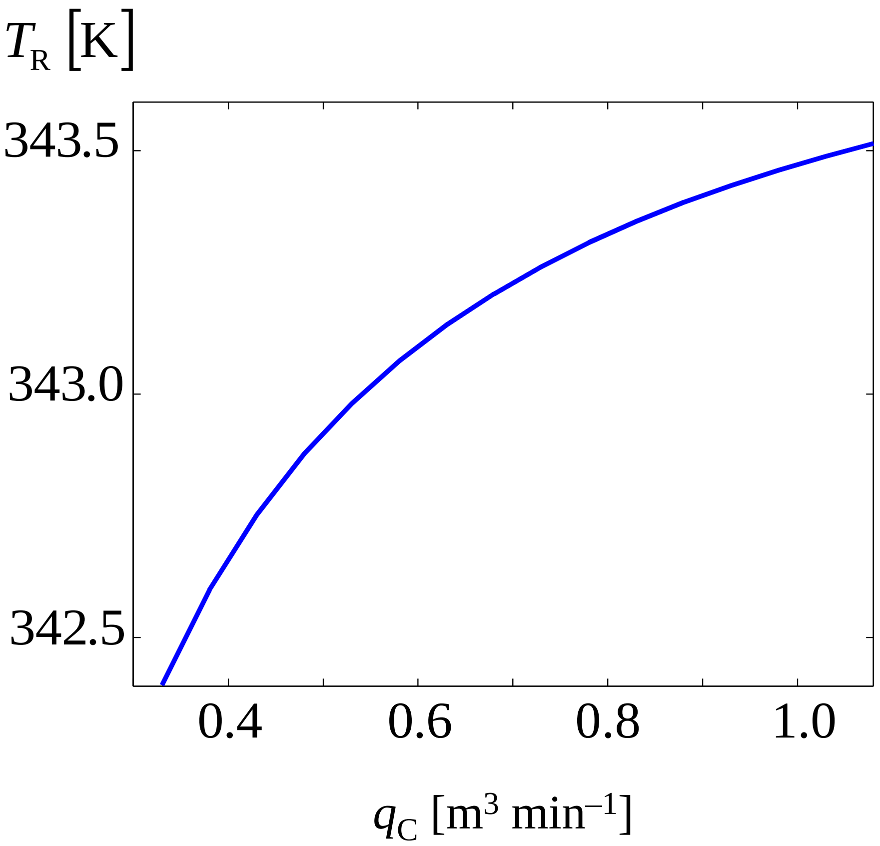


Static Characteristics

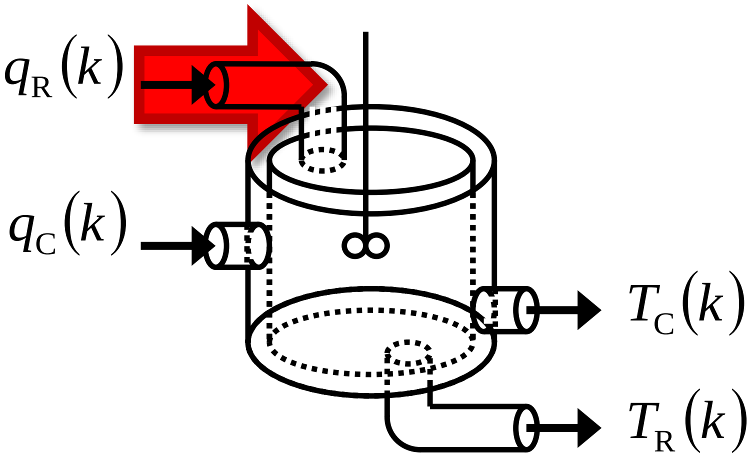


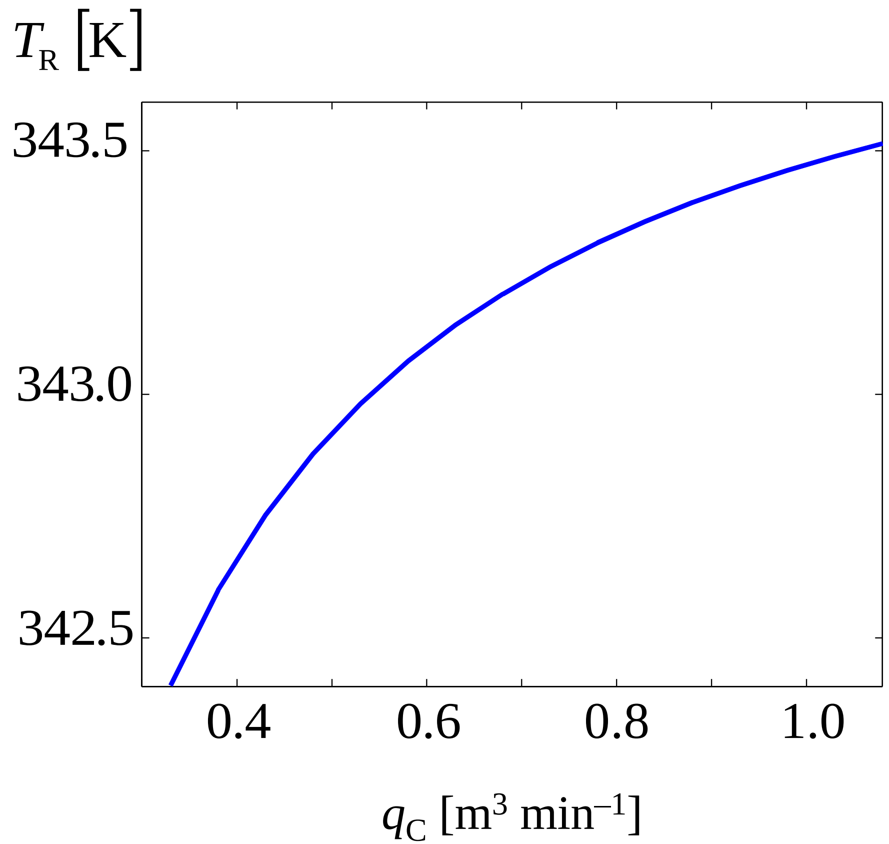
Static Characteristics



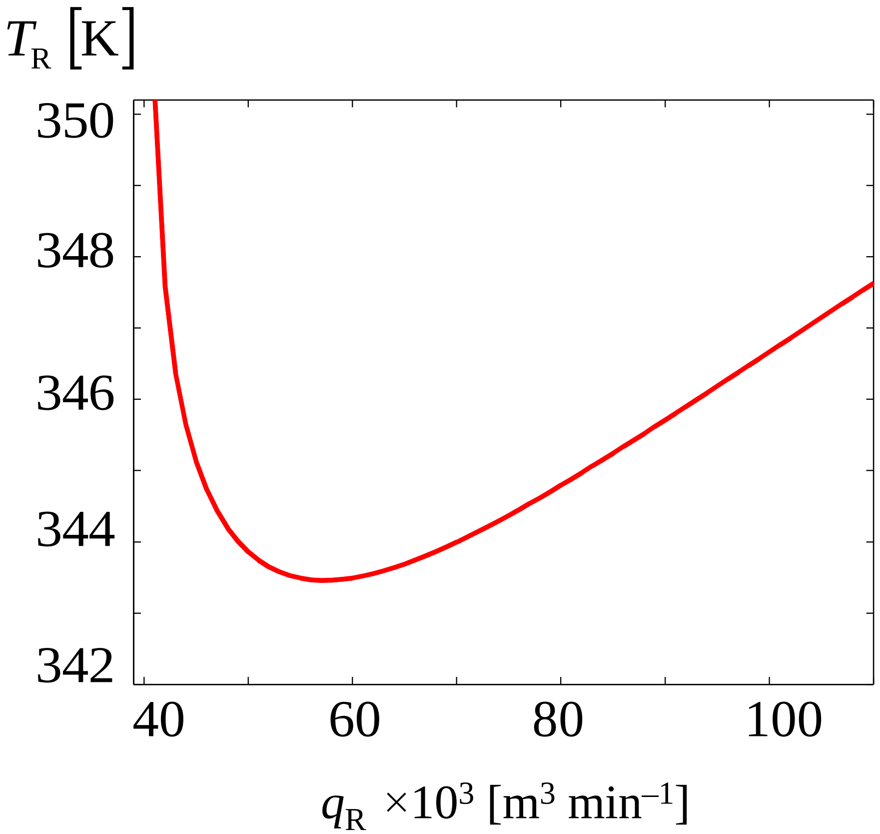


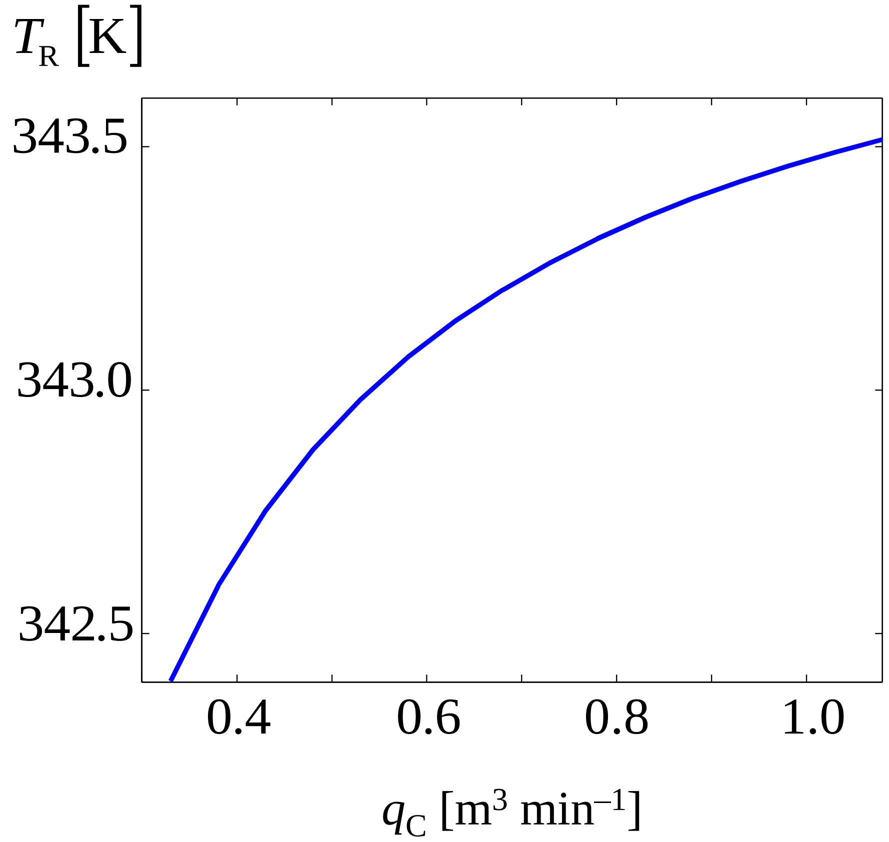
Static Characteristics



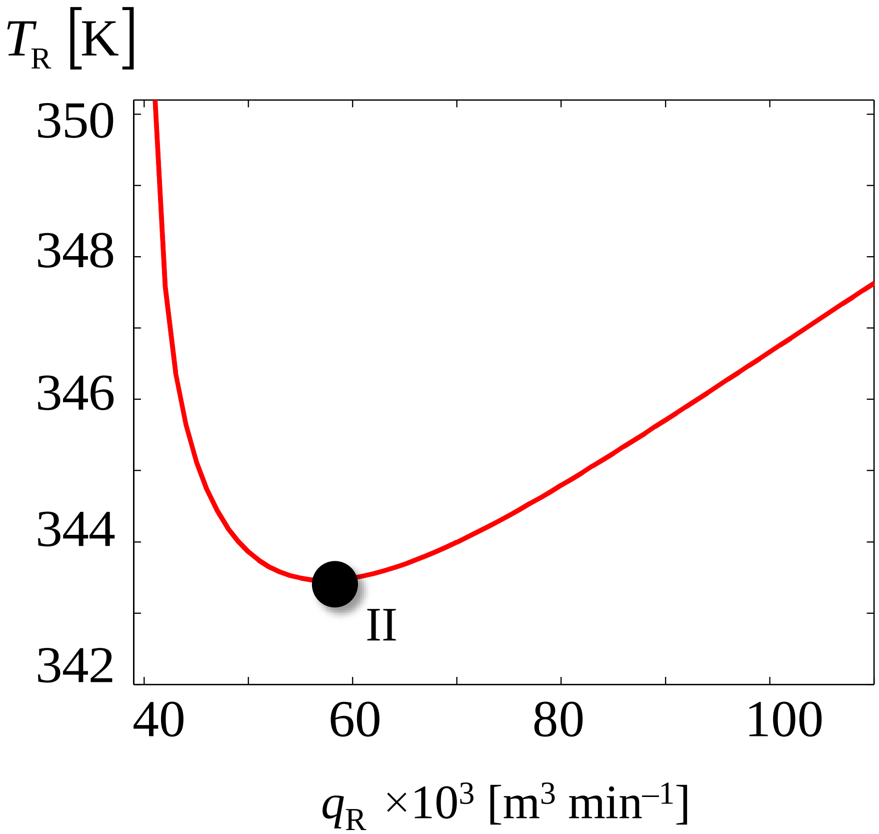


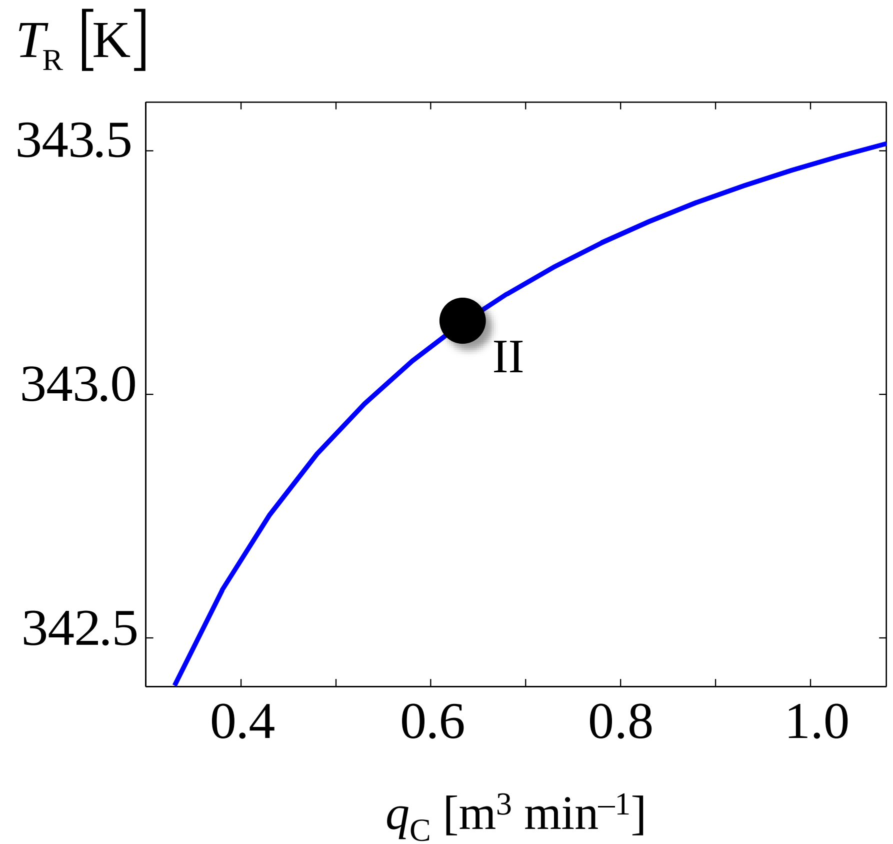
Static Characteristics



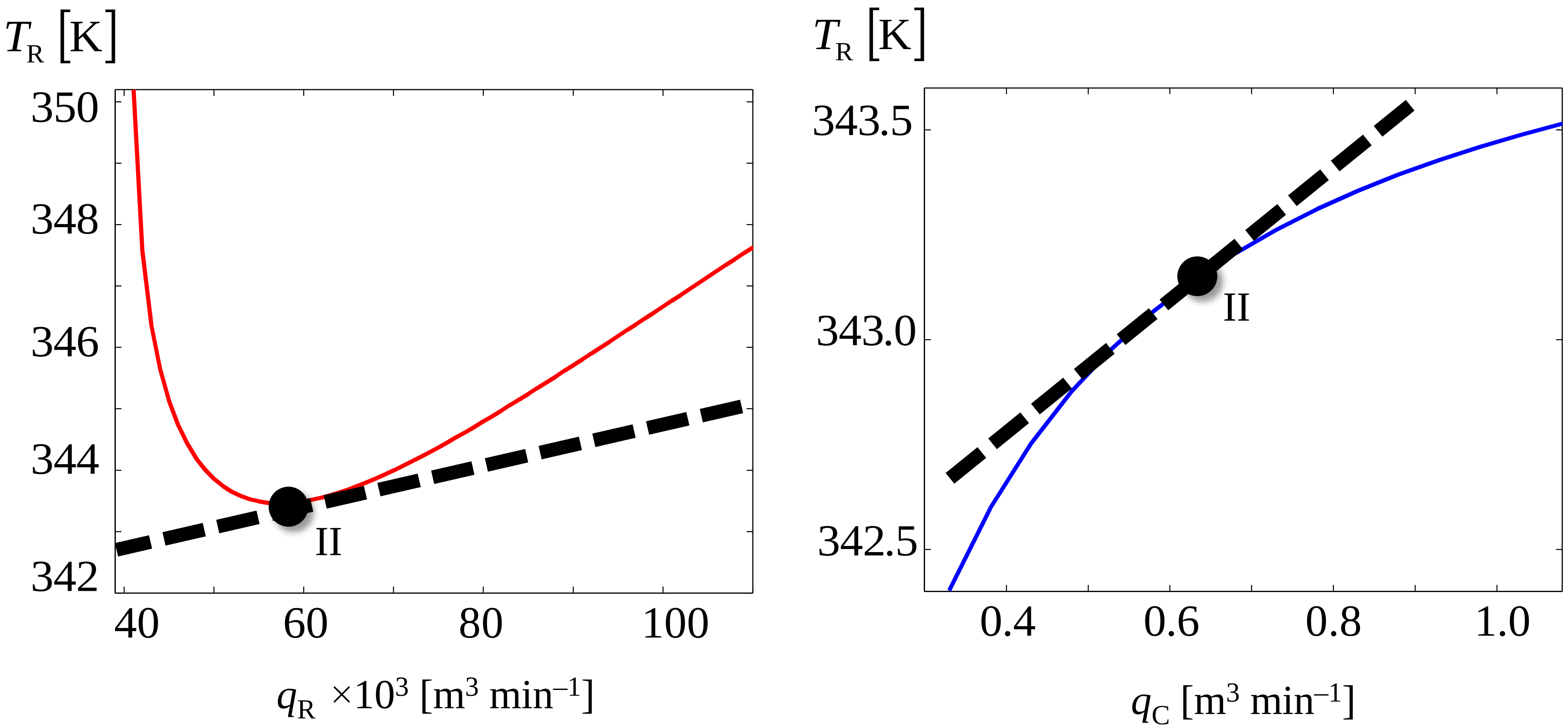


Static Characteristics

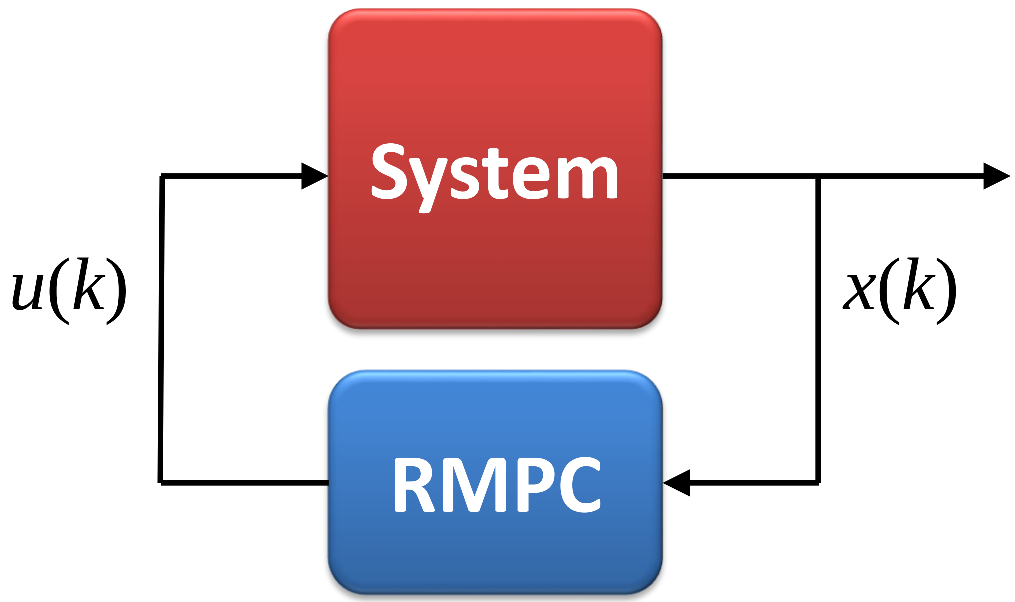




Static Characteristics

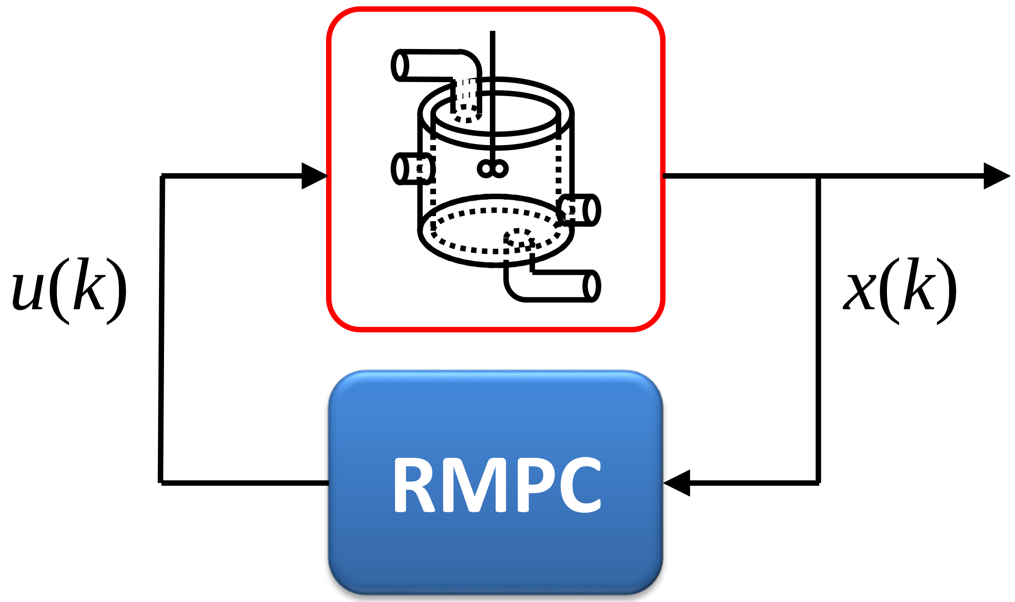


Control of CSTR



$$x(k+1) = A\,x(k) + B\,u(k), \qquad x(0) = x_0,$$
$$y(k) = C\,x(k), \qquad y \in \mathbb{Y},$$
$$[A, B] \in \text{convhull}\{[A_v, B_v]\}, \qquad u \in \mathbb{U}.$$

Control of CSTR



$$x(k+1) = A\,x(k) + B\,u(k), \qquad x(0) = x_0,$$
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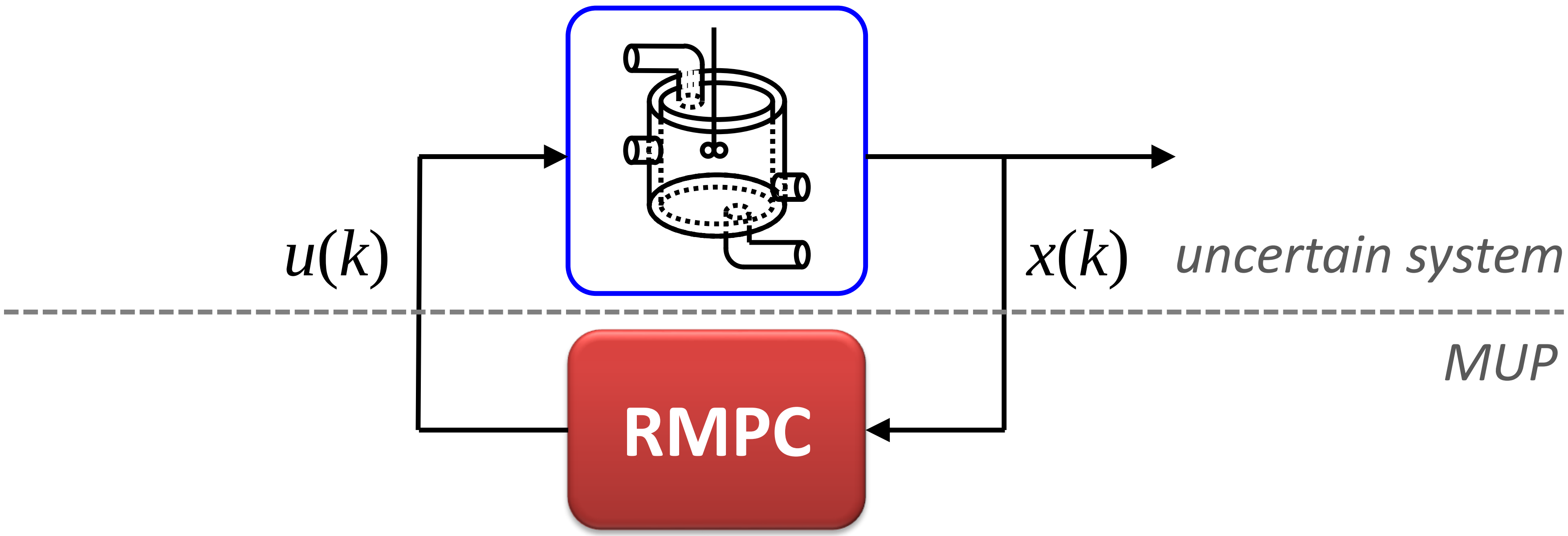
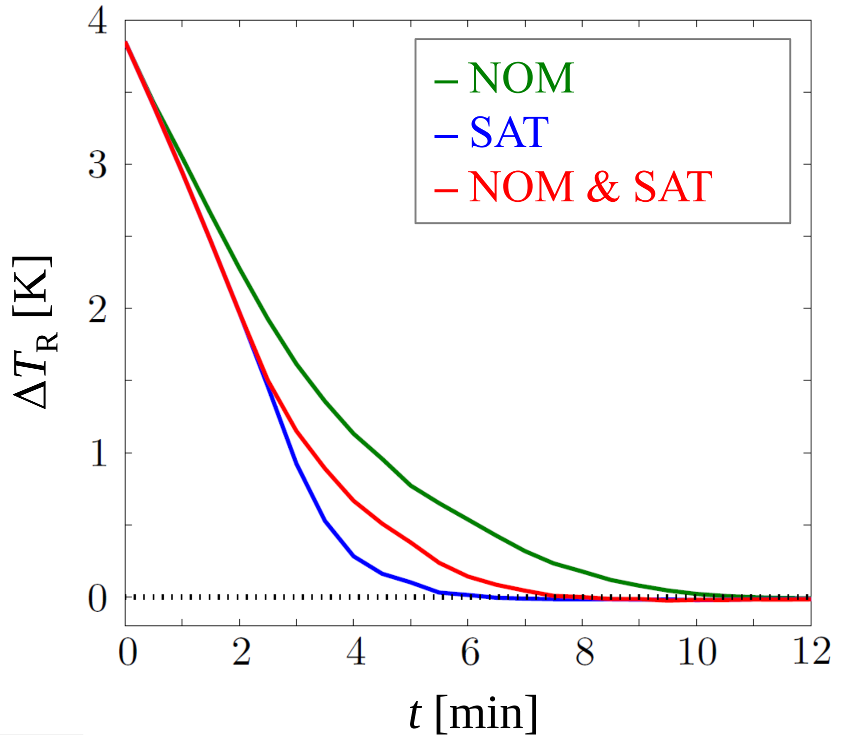
Control of CSTR



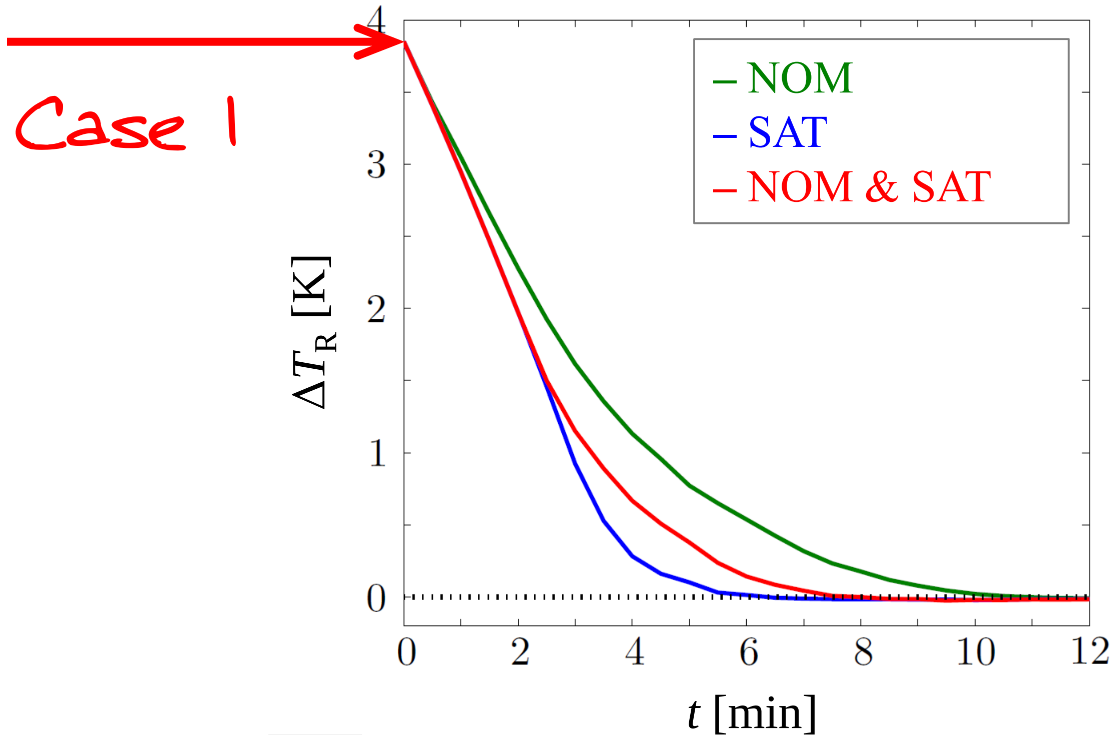
MATLAB toolbox for on-line RMPC design by LMIs

★ *bitbucket.org/oravec/mup*

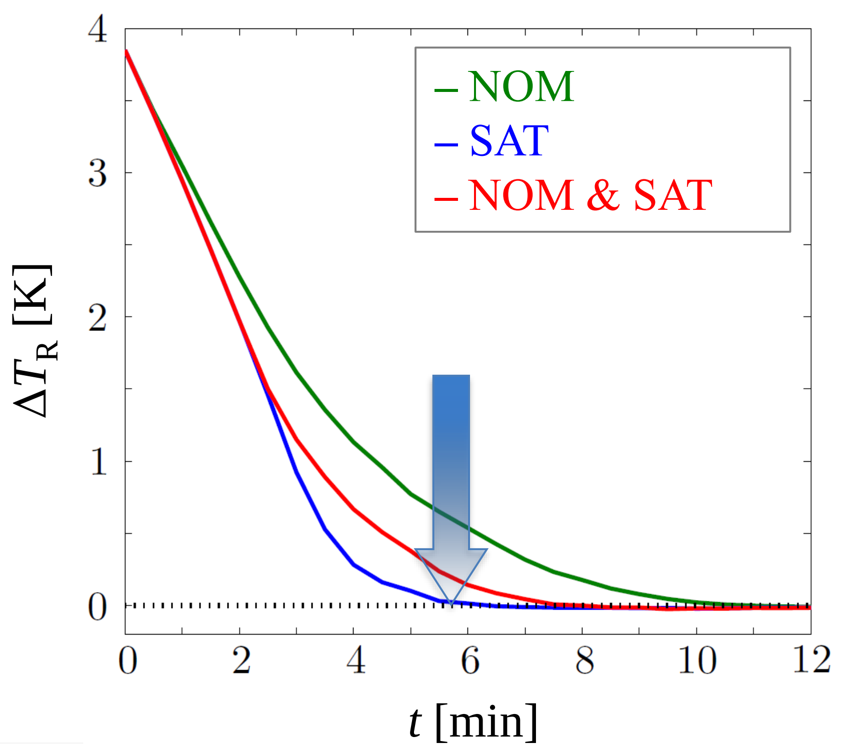
Control of CSTR - Case 1



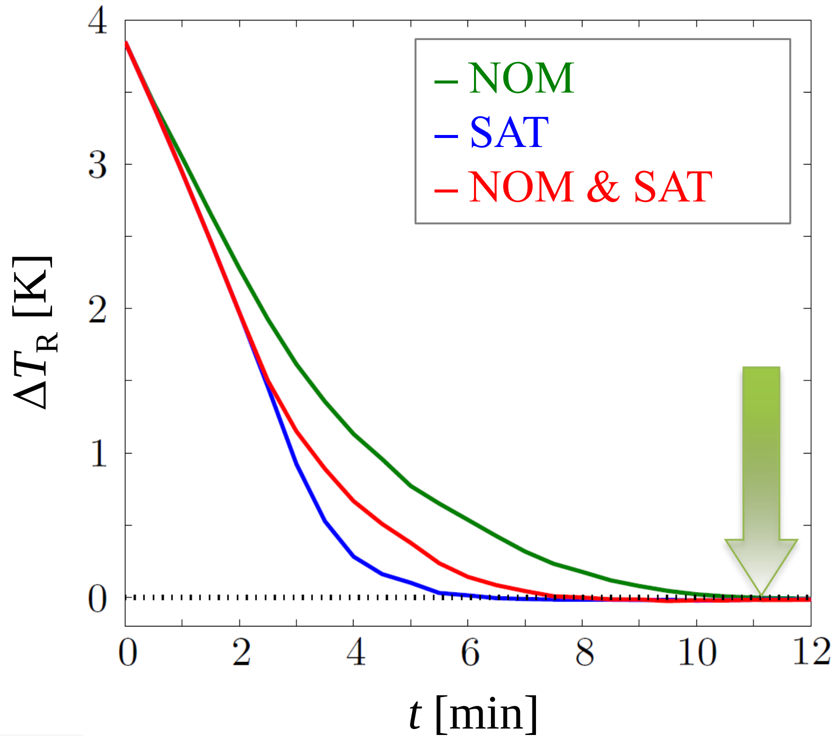
Control of CSTR - Case 1



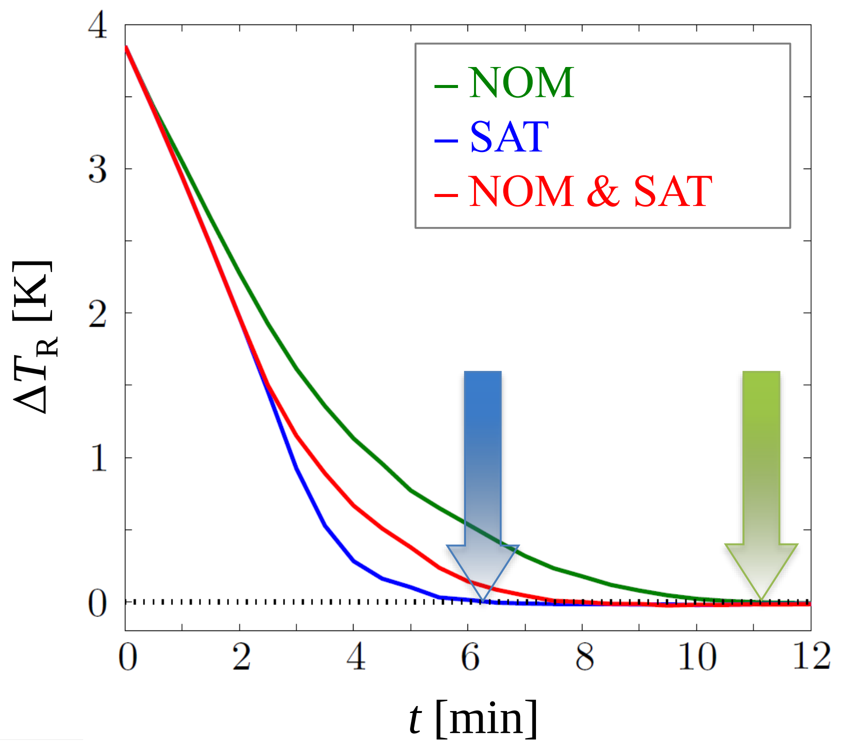
Control of CSTR - Case 1



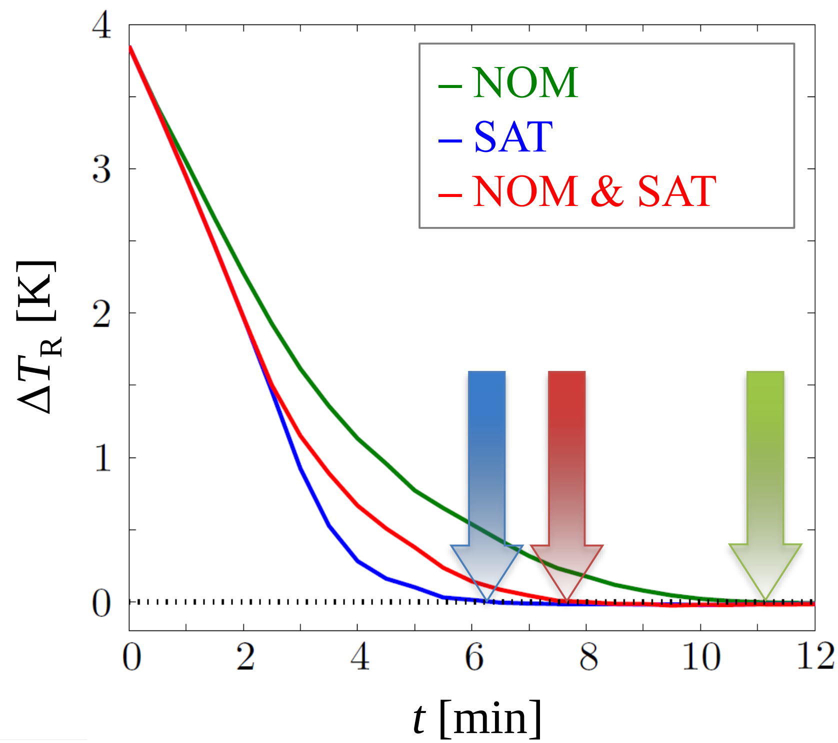
Control of CSTR - Case 1



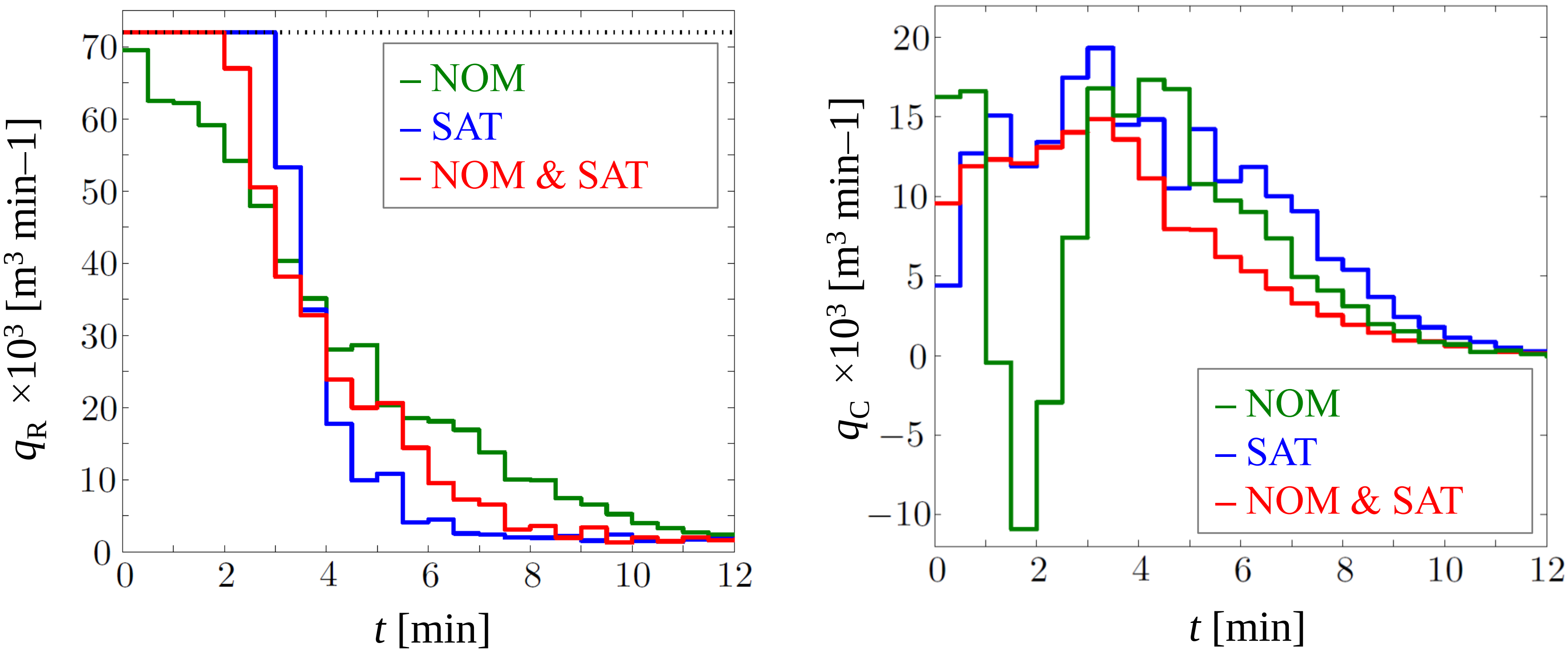
Control of CSTR - Case 1



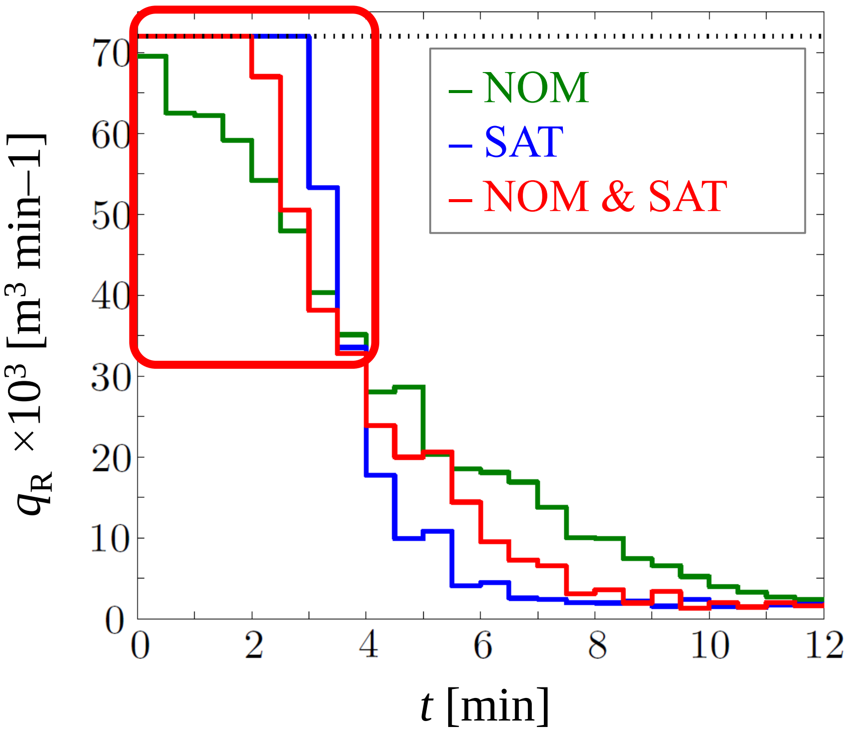
Control of CSTR - Case 1

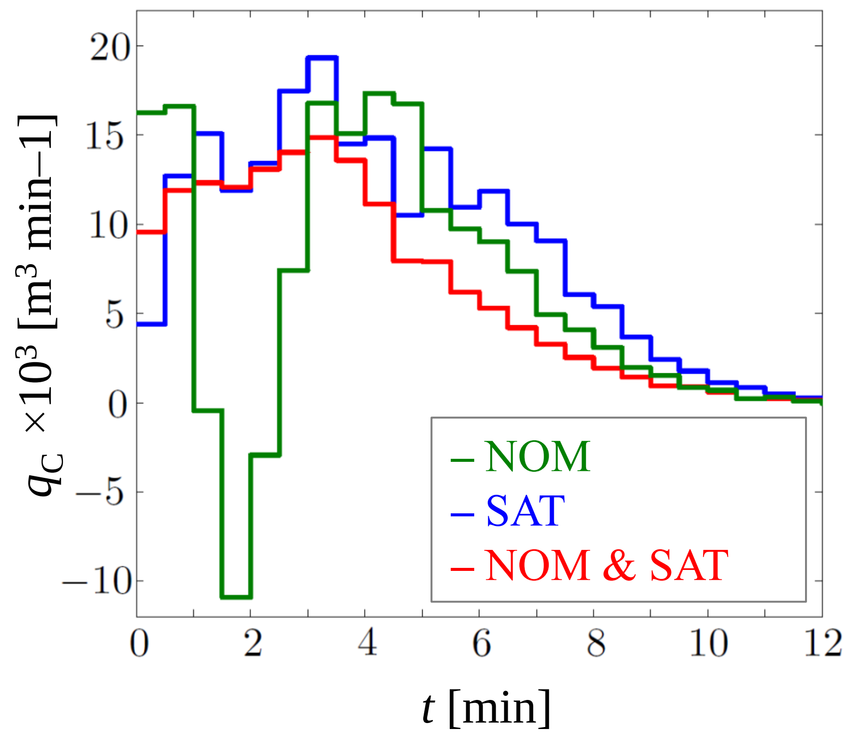


Control of CSTR - Case I

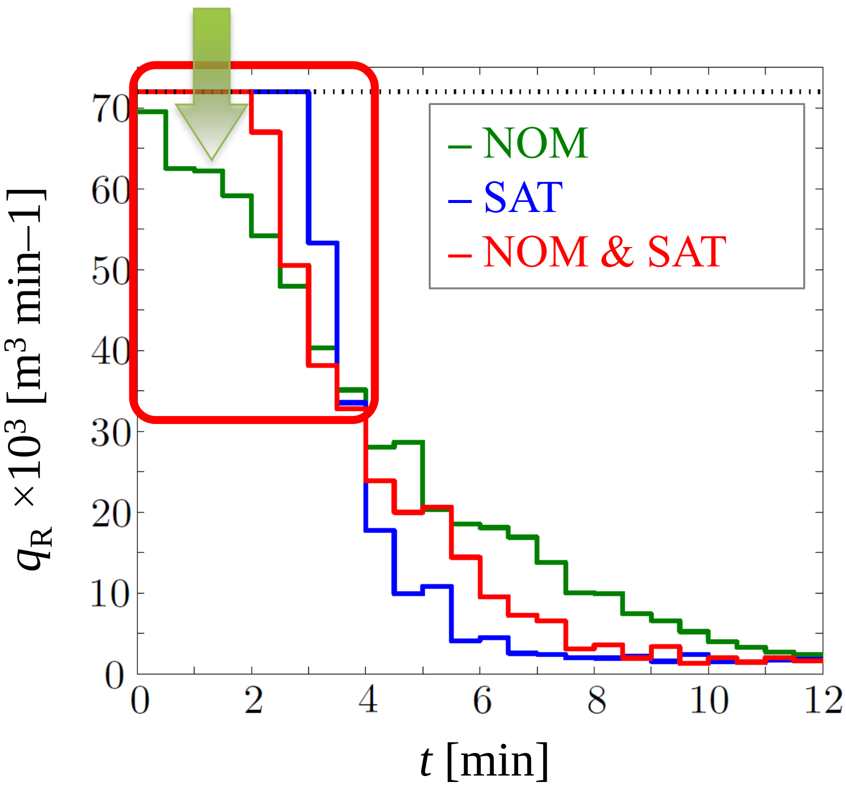


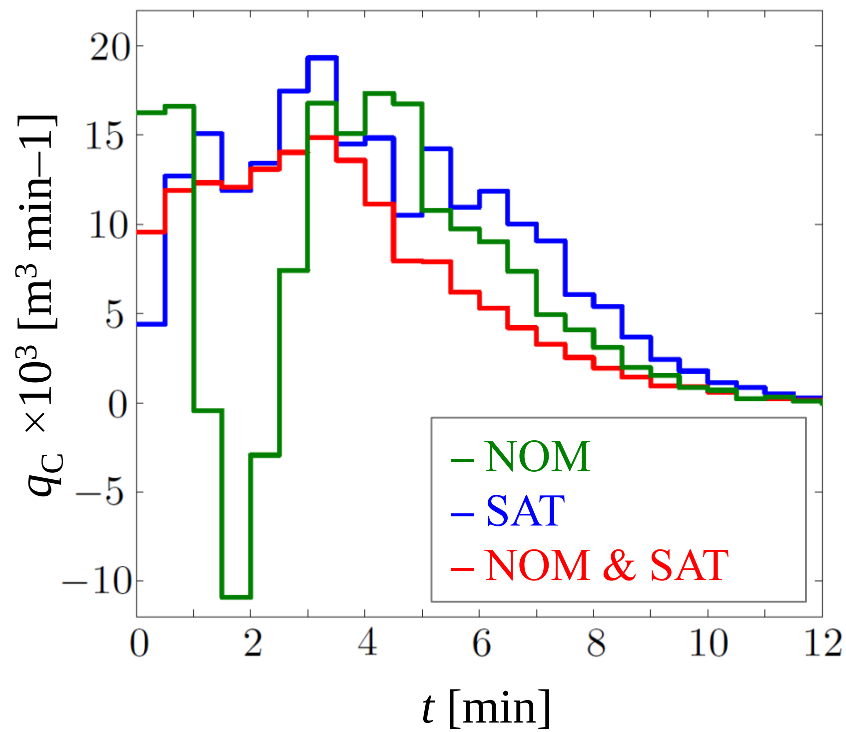
Control of CSTR - Case I



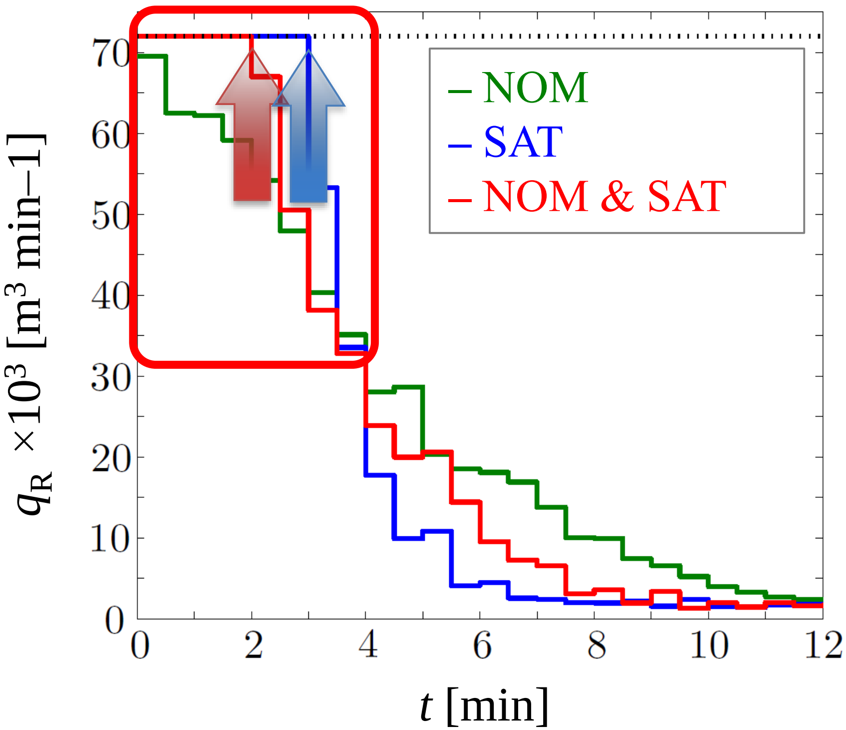


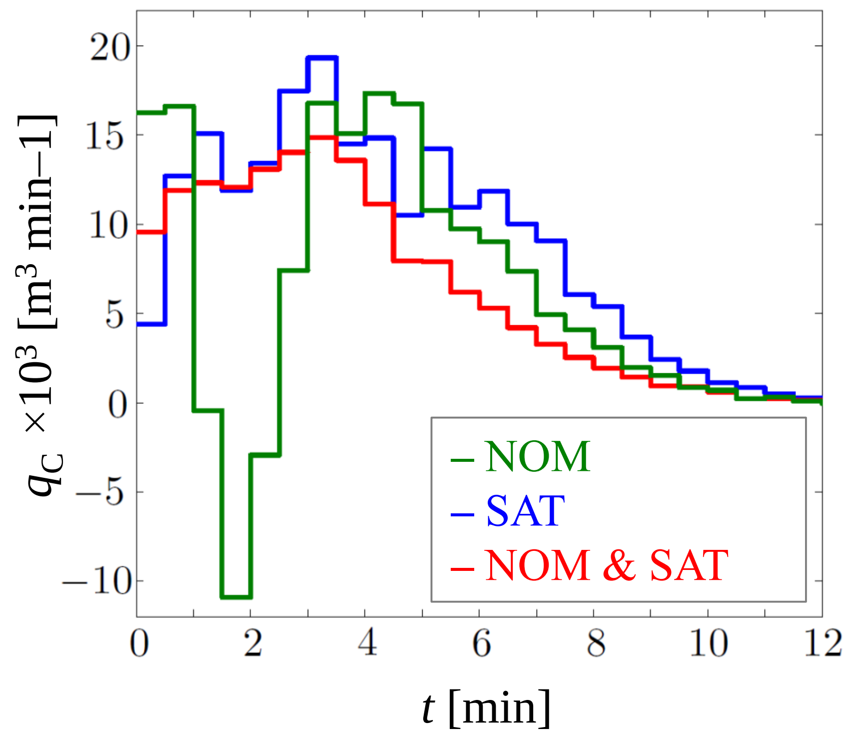
Control of CSTR - Case I



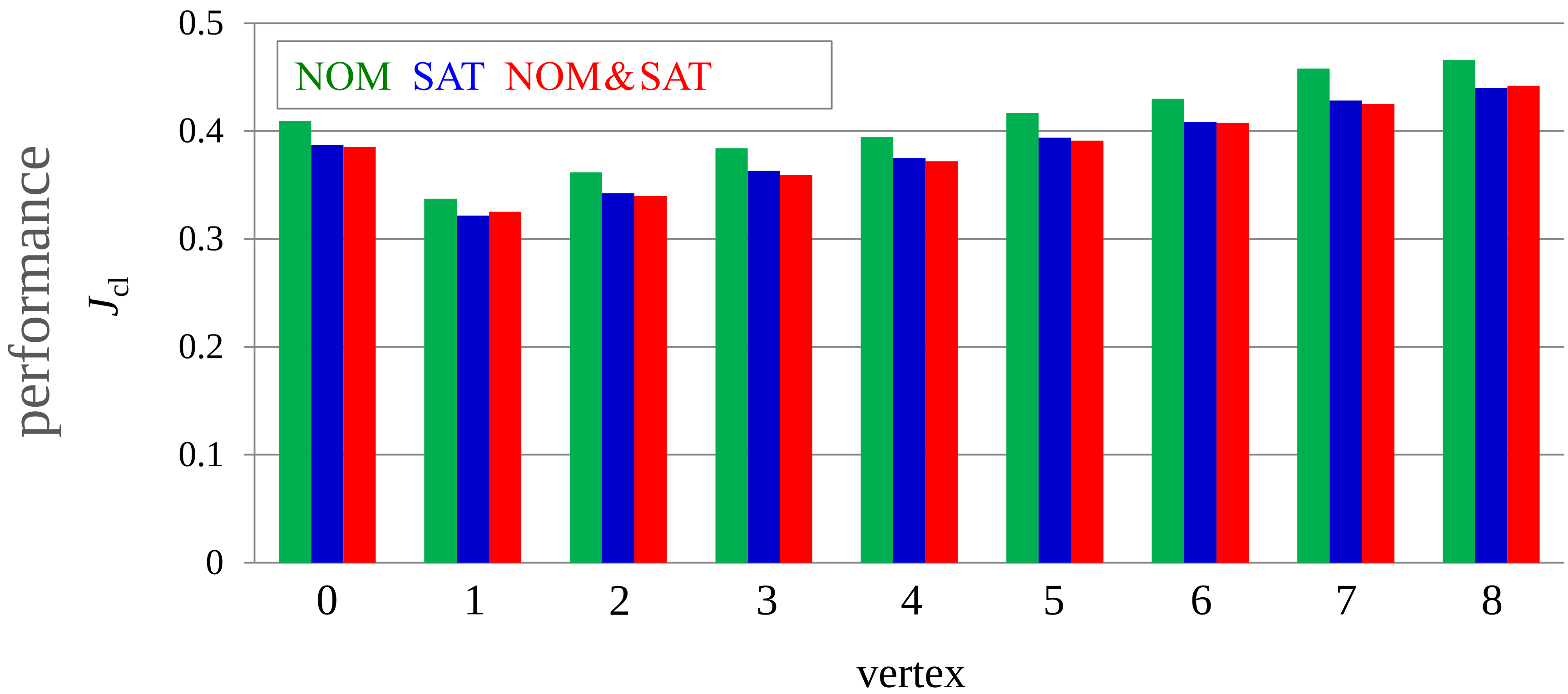


Control of CSTR - Case 1



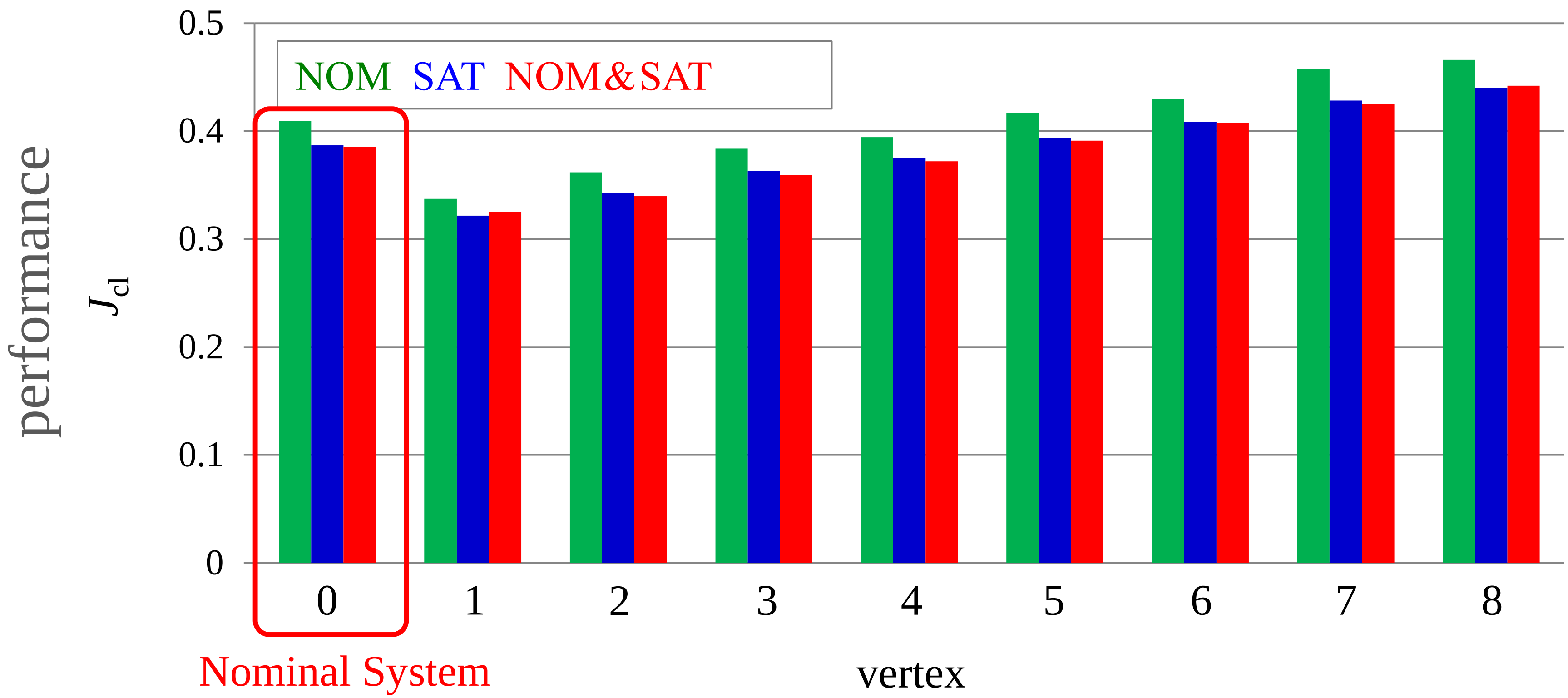


Control of CSTR - Case 1



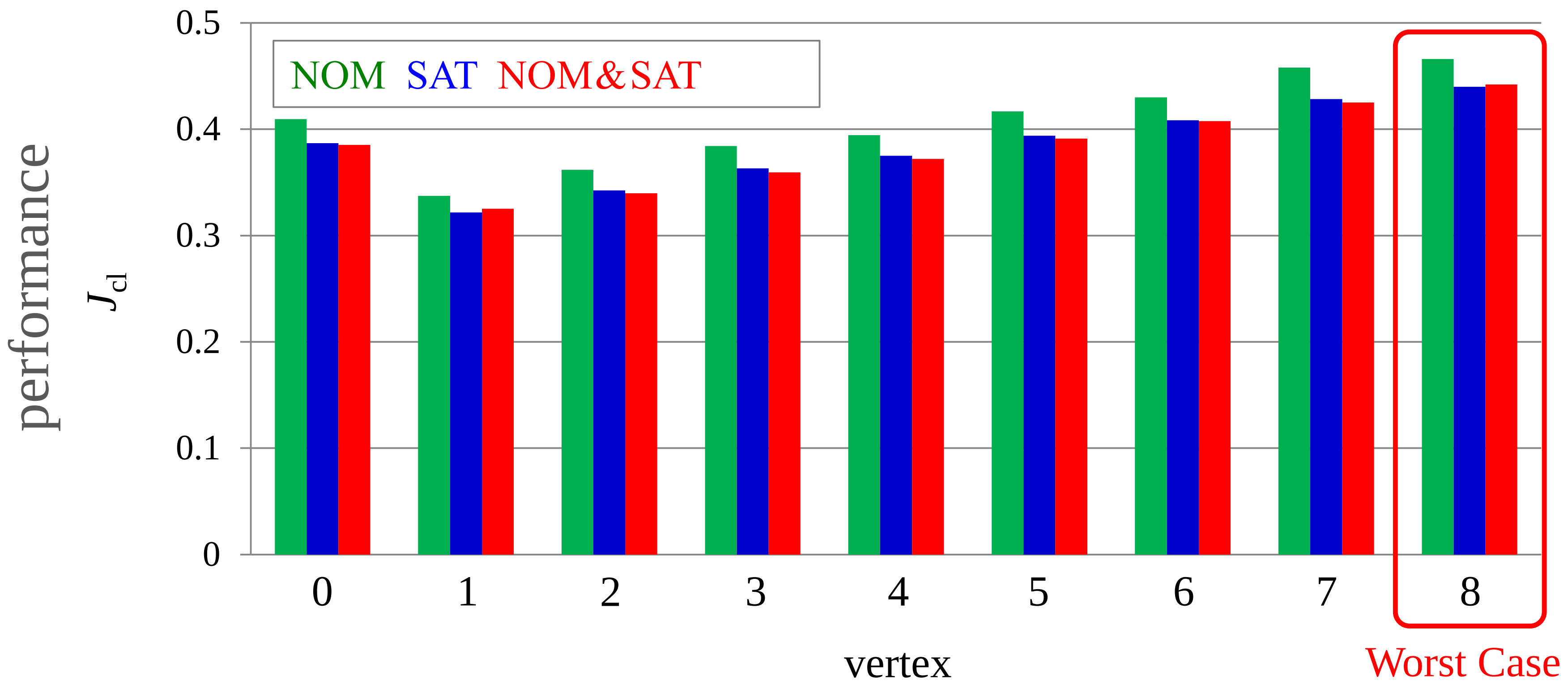
Approach	
NOM	
SAT	
NOM & SAT	

Control of CSTR - Case 1



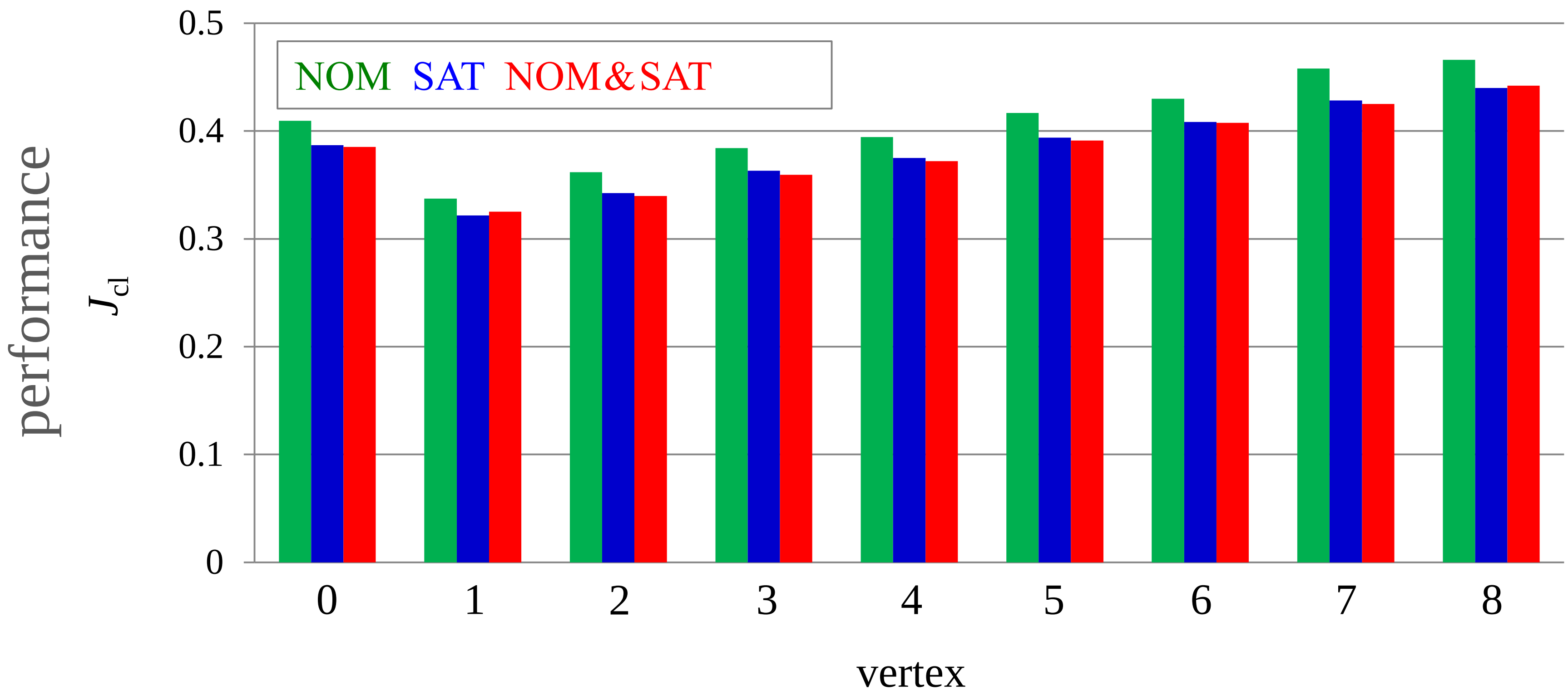
Approach	Nominal System [%]
NOM	0
SAT	–6.0 %
NOM & SAT	–5.5 %

Control of CSTR - Case 1



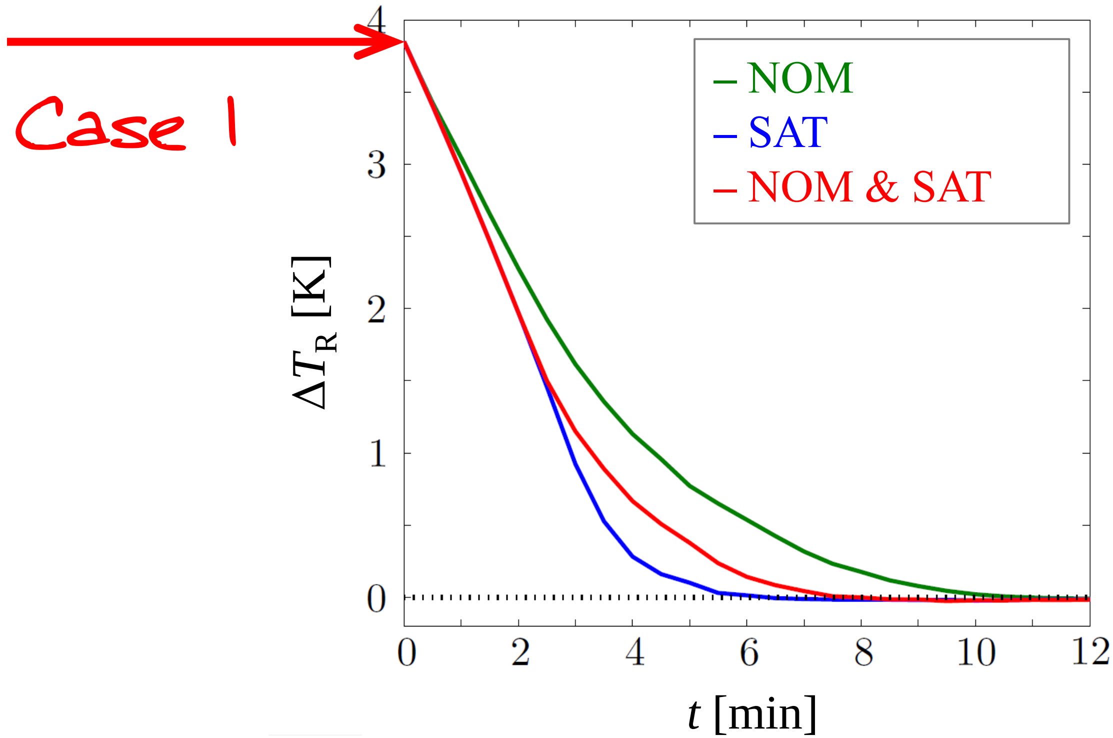
Approach	Nominal System [%]	Worst Case [%]
NOM	0	0
SAT	–6.0 %	–5.5 %
NOM & SAT	–5.5 %	–5.0 %

Control of CSTR - Case I

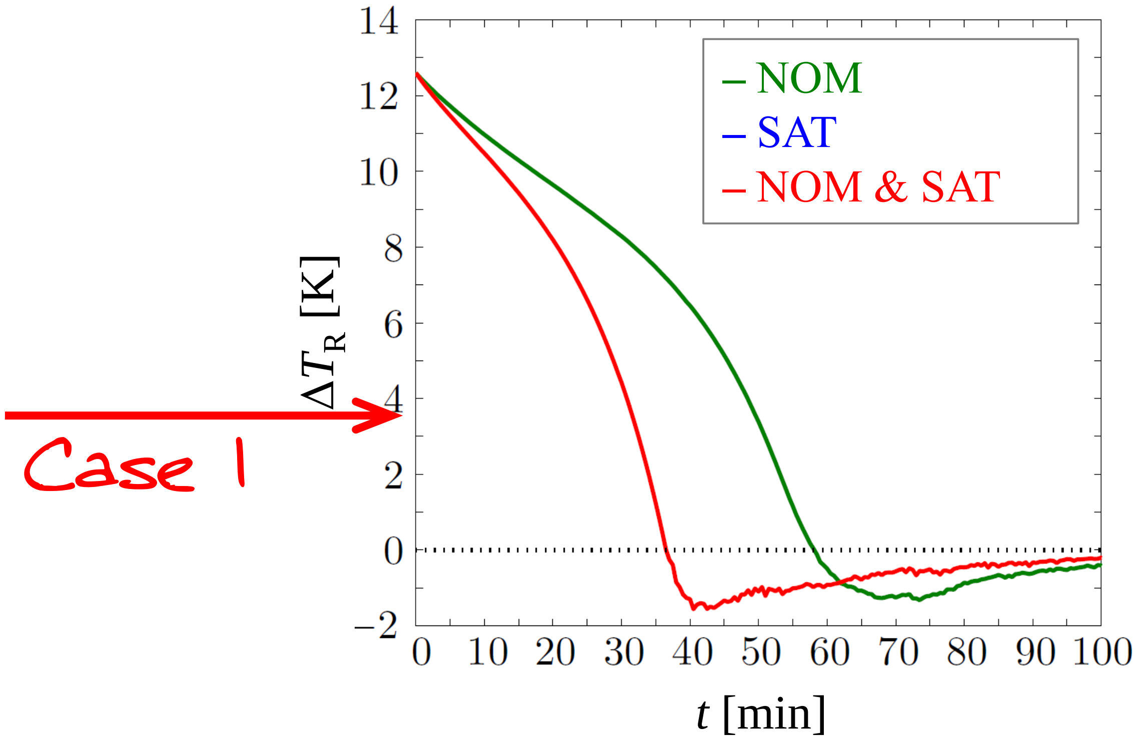


Approach	Nominal System [%]	Worst Case [%]	t_{sol} [s]
NOM	0	0	1.2
SAT	–6.0 %	–5.5 %	5.6
NOM & SAT	–5.5 %	–5.0 %	3.2

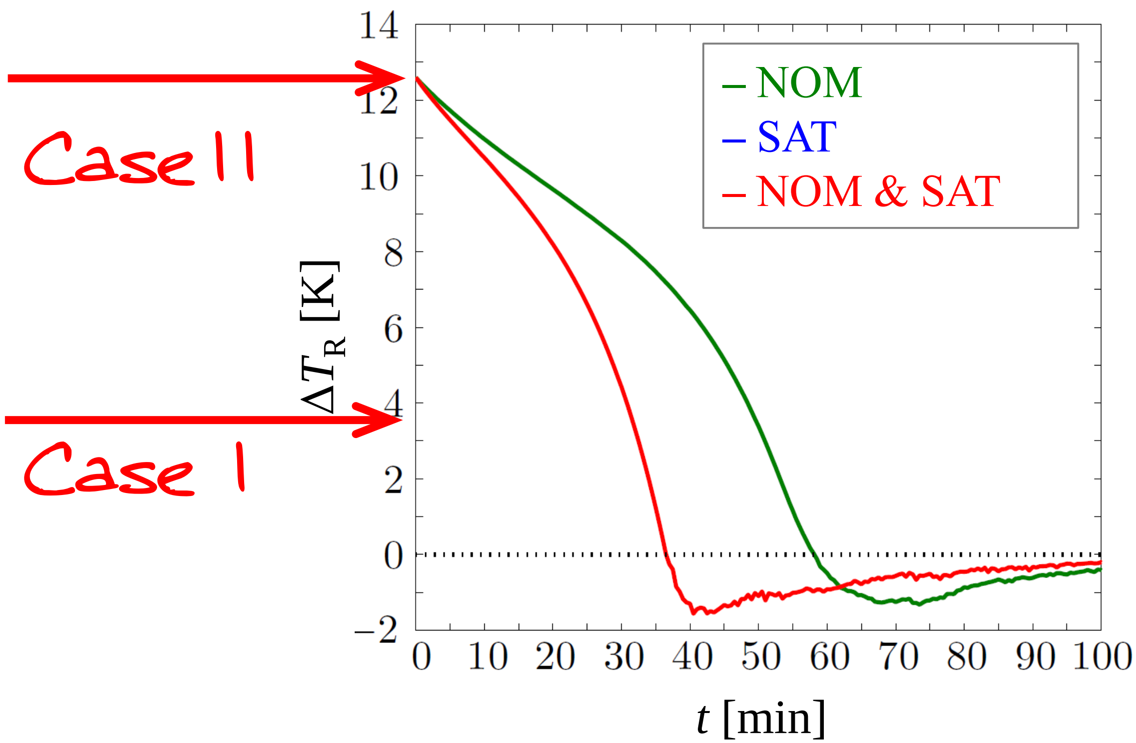
Control of CSTR - Case 1



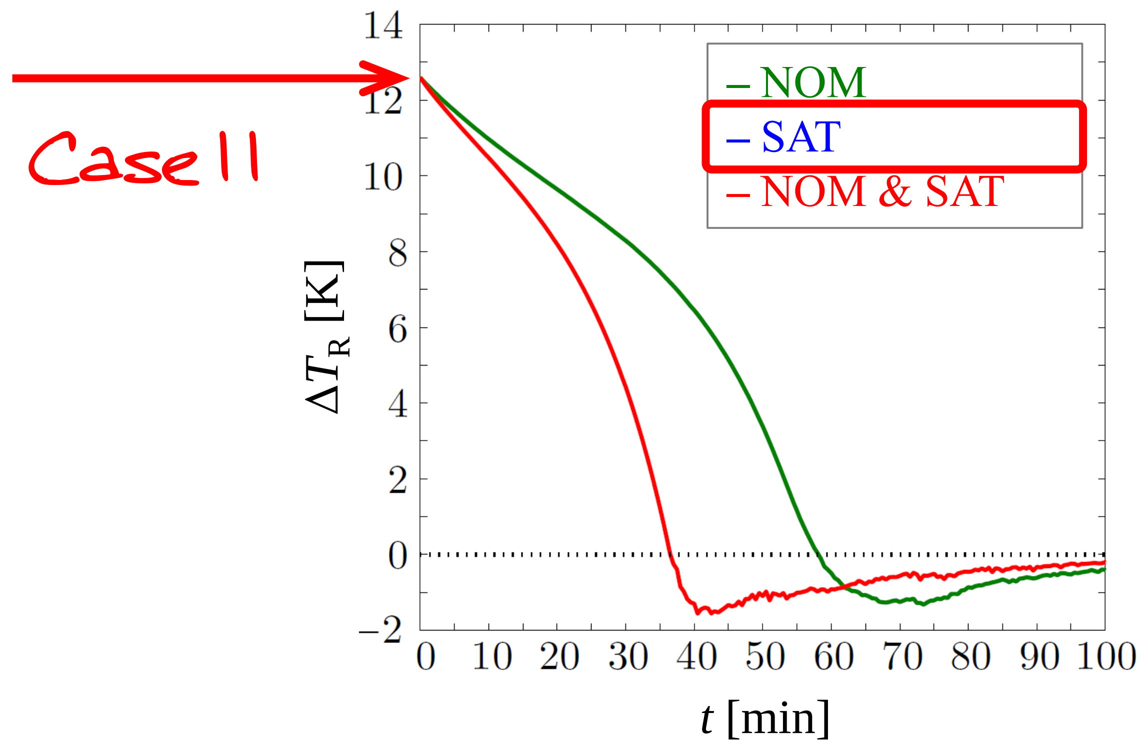
Control of CSTR - Case II



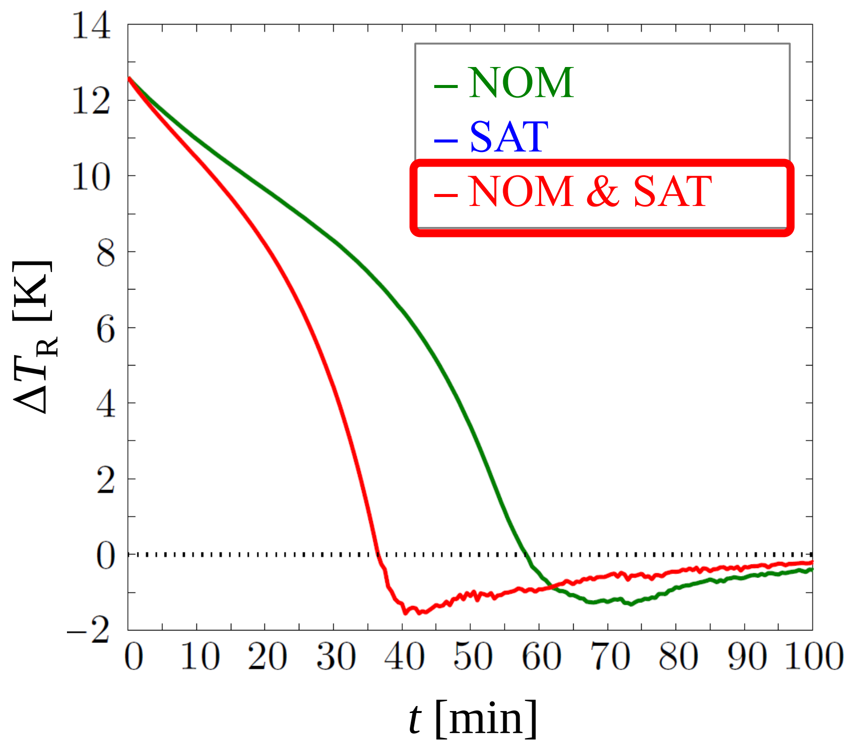
Control of CSTR - Case II



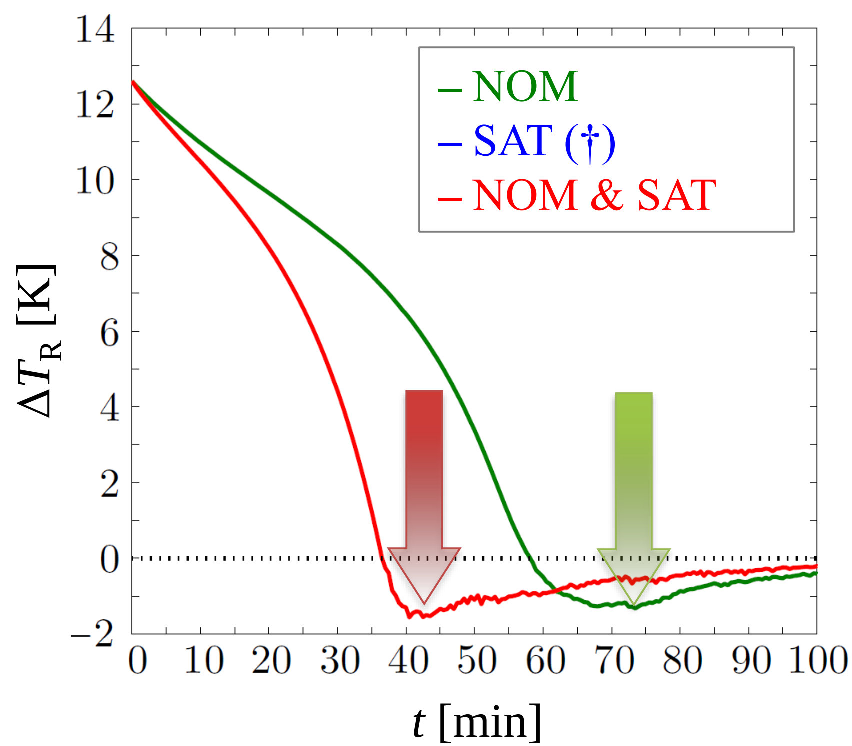
Control of CSTR - Case II



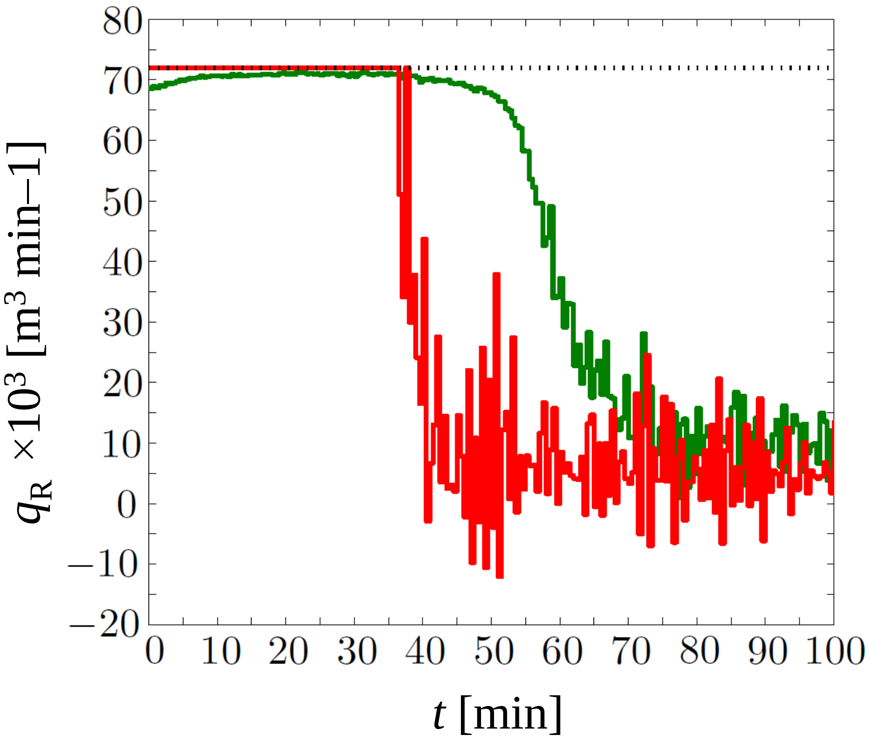
Control of CSTR - Case II

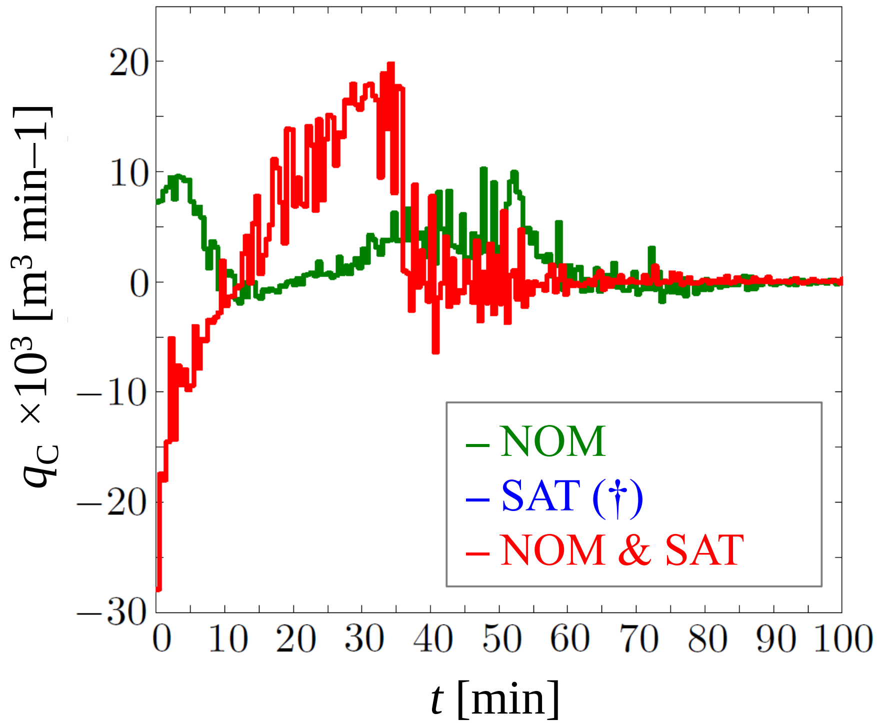


Control of CSTR - Case II

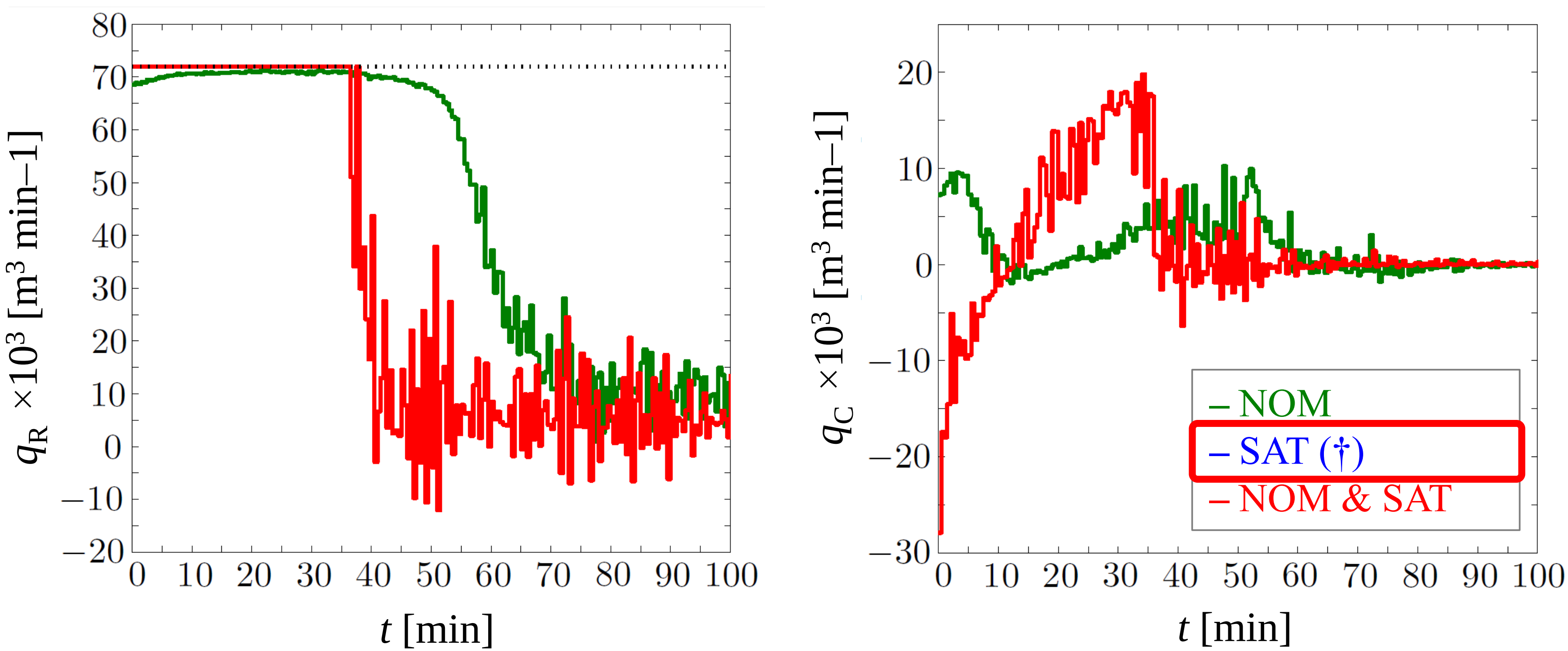


Control of CSTR - Case II

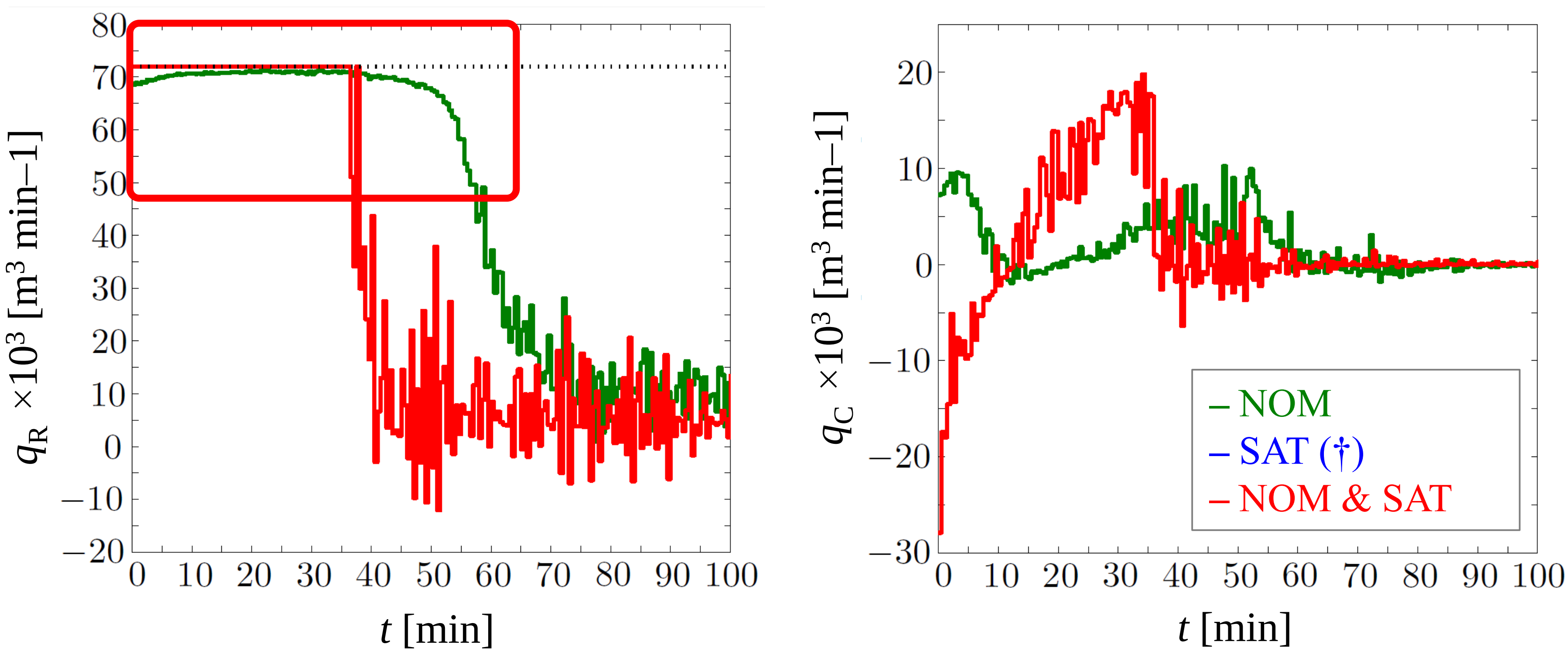




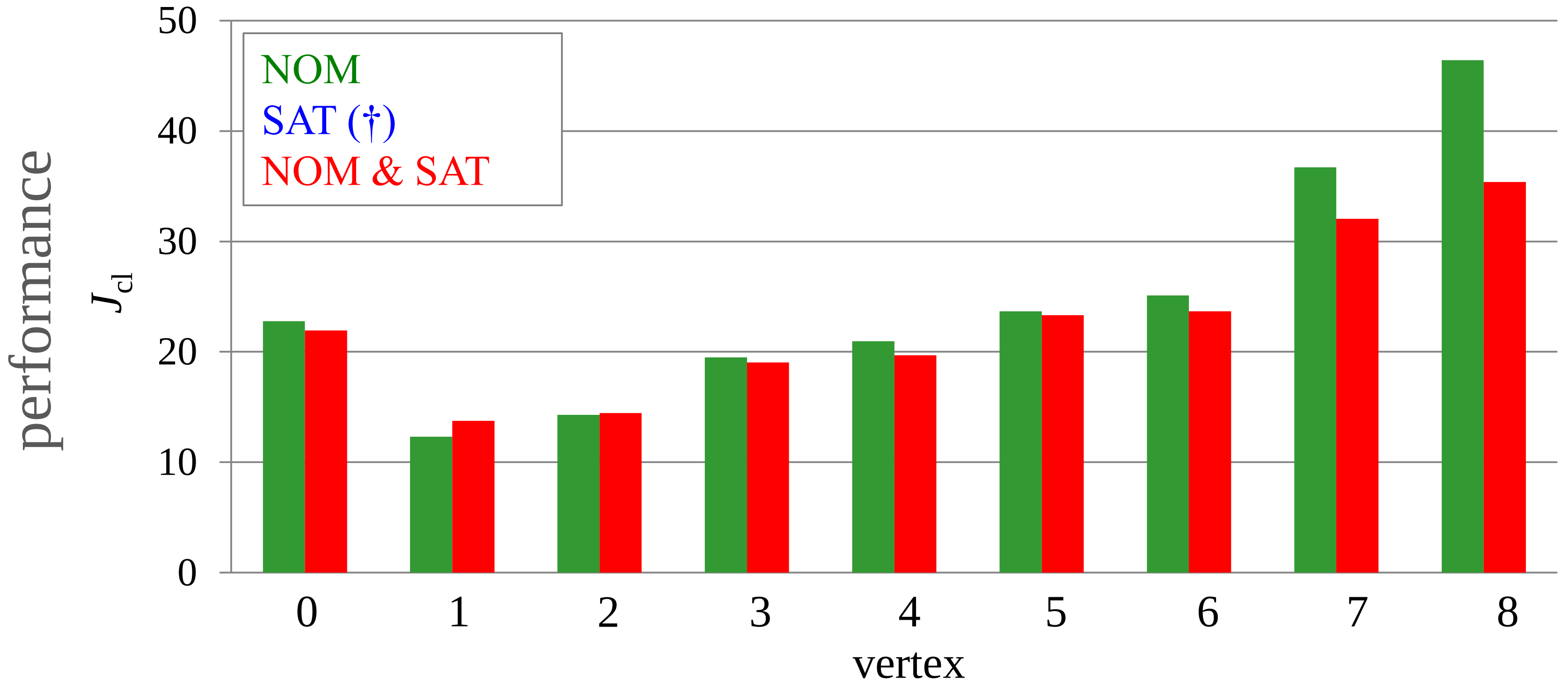
Control of CSTR - Case II



Control of CSTR - Case II

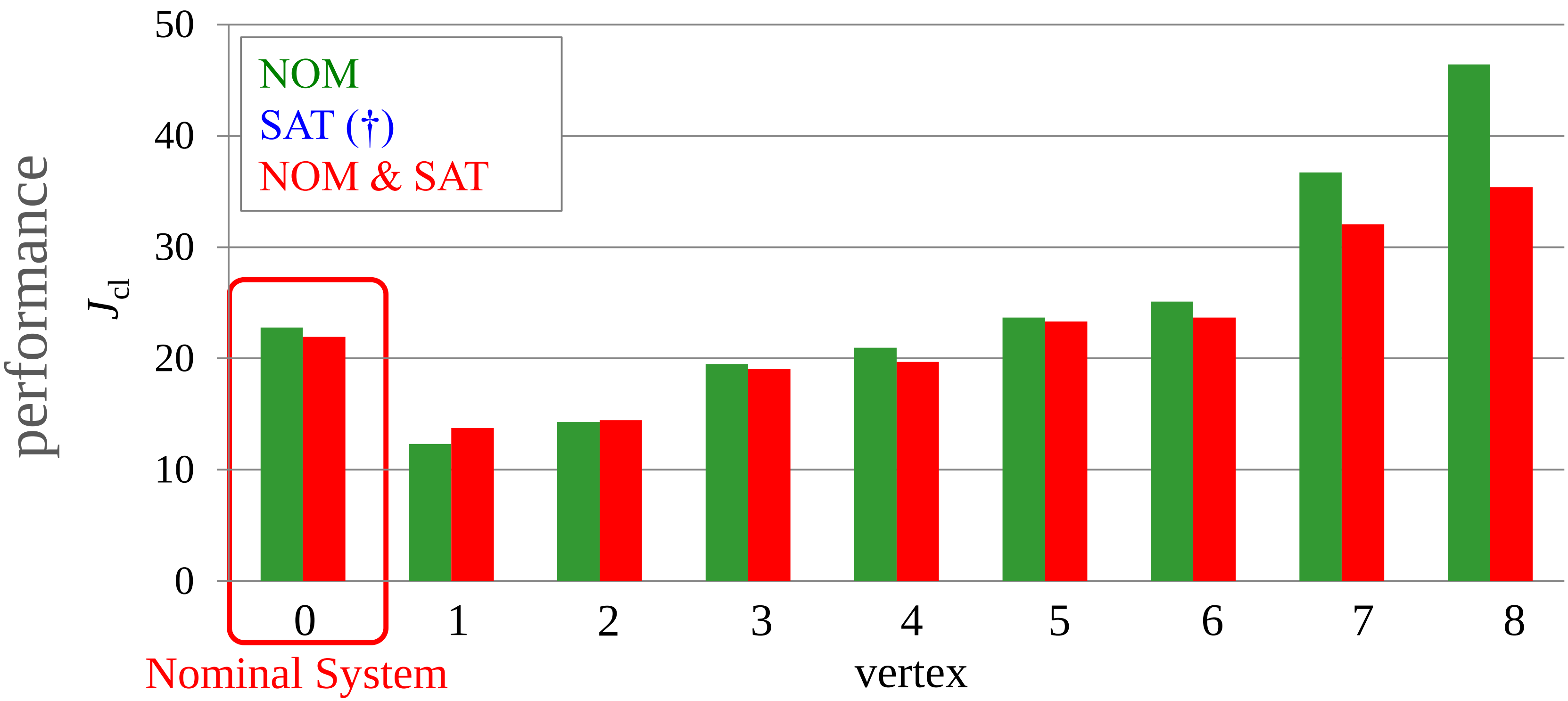


Control of CSTR - Case II



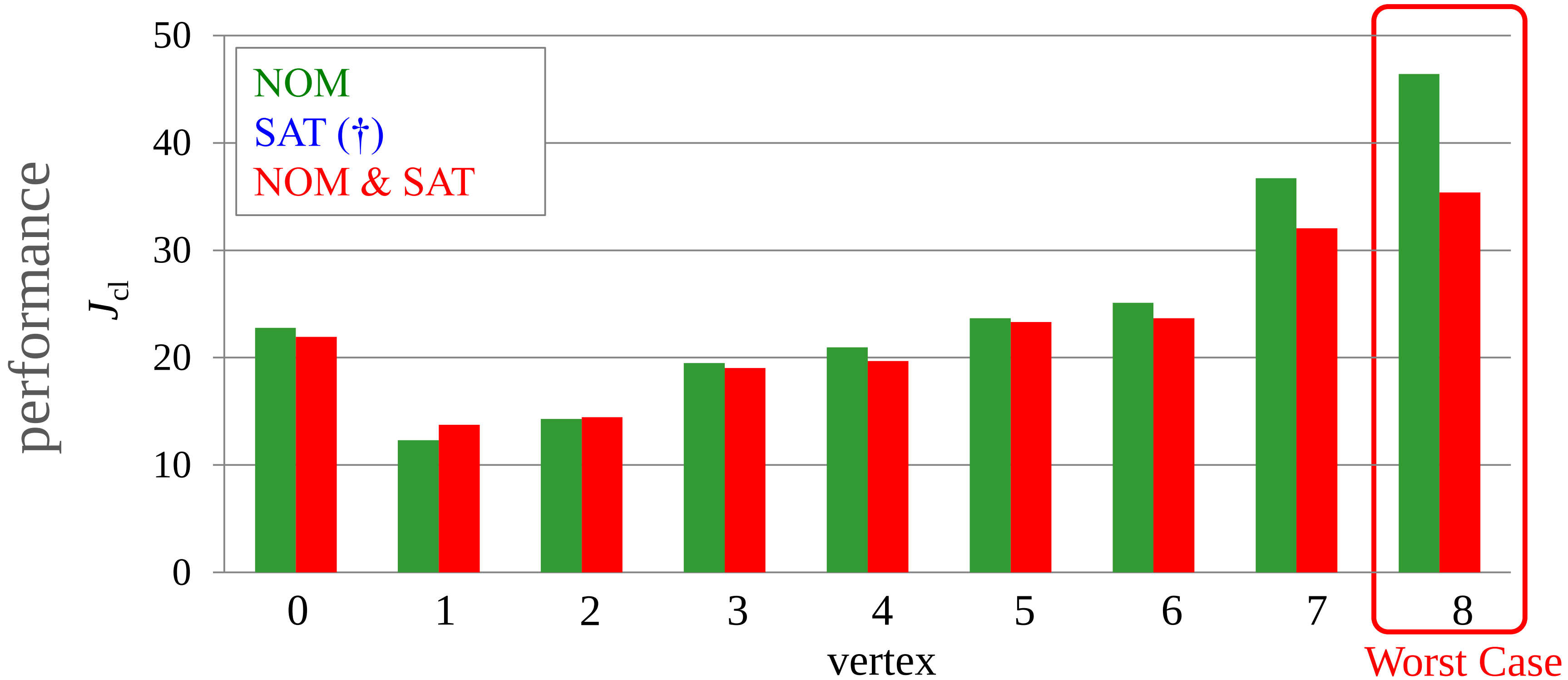
Approach	
NOM	
SAT	
NOM & SAT	

Control of CSTR - Case II



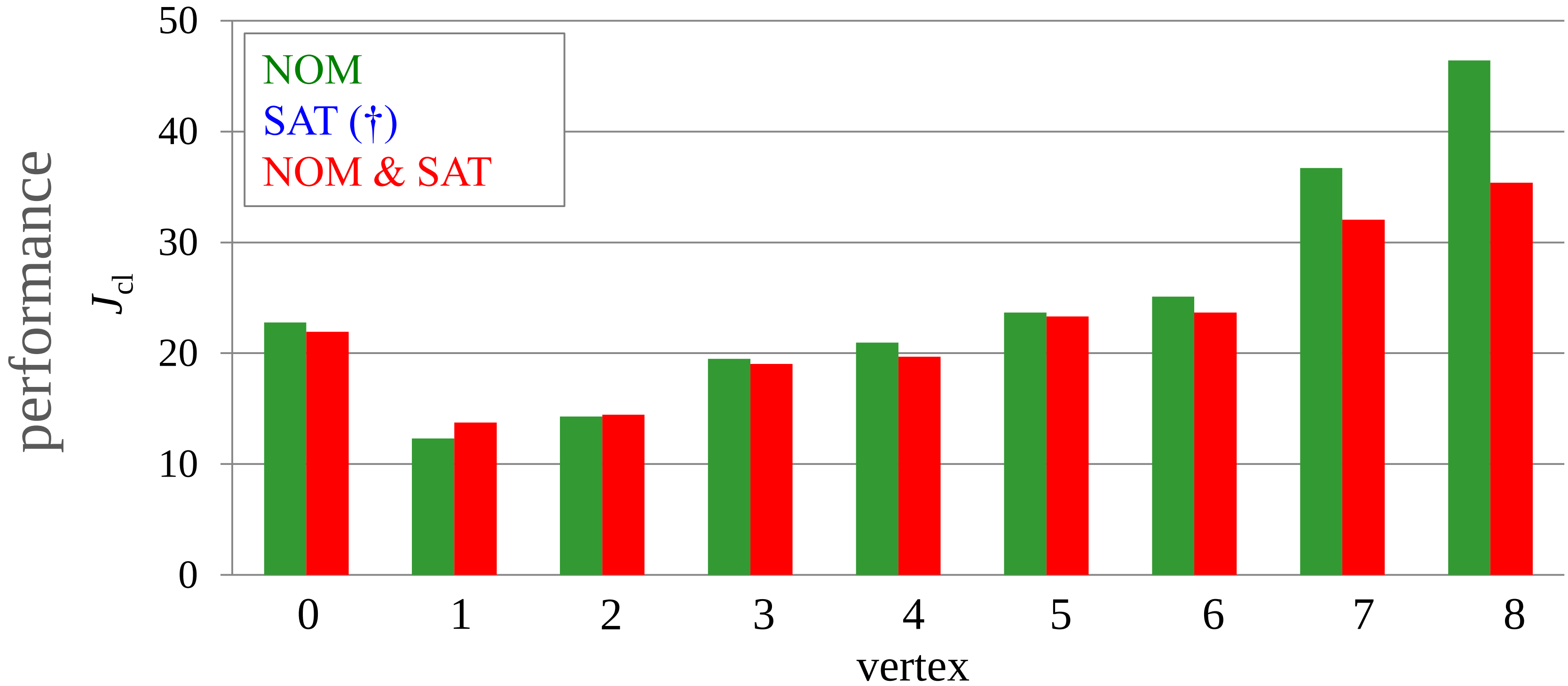
Approach	Nominal System [%]
NOM	0
SAT	†
NOM & SAT	–4.0

Control of CSTR - Case II



Approach	Nominal System [%]	Worst Case [%]	
NOM	0	0	
SAT	†	†	
NOM & SAT	–4.0	–24.0	

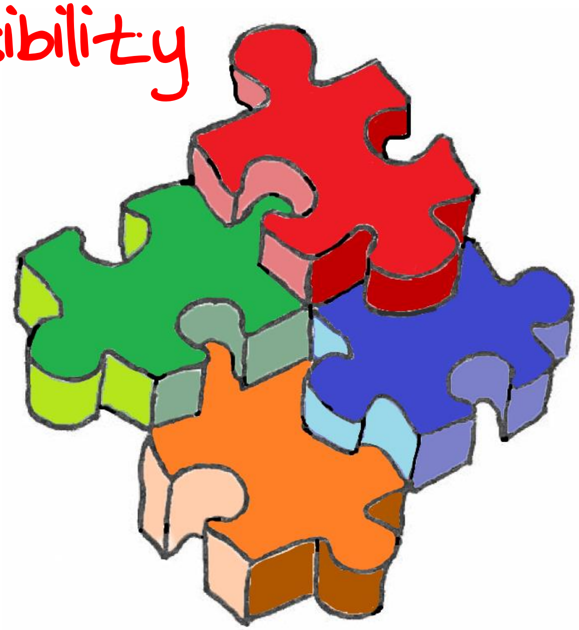
Control of CSTR - Case II



Approach	Nominal System [%]	Worst Case [%]	t_{sol} [s]
NOM	0	0	1.2
SAT	†	†	†
NOM & SAT	–4.0	–24.0	3.2

Alternative Approaches

feasibility

performance  efficiency

conservativeness

MATLAB toolbox for on-line RMPC design by LMIs

* *bitbucket.org/oravec/mup*

References

[1] *Z. Wan, M. V. Kothare* (2003)*: Efficient Robust Constrained Model Predictive Control with a Time Varying Terminal Constraint Set. System & Control Letters* 48, 375-383*.*

[2] *Y. Y. Cao, Z. Lin* (2005)*: Min-max MPC Algorithm for LPV Systems Subject to Input Saturation. Control Theory and Applications, IEE Proceedings* 153, 266-272*.*

[3] *H. Huang, D. Li, Z. Lin, Y. Xi* (2011)*: An Improved Robust Model Predictive Control Design in the Presence of Actuator Saturation. Automatica* 47, 861-864*.*

[4] *W. J. Mao* (2003)*: Robust Stabilization of Uncertain Time-Varying Discrete Systems and Comments on "An Improved Approach for Constrained Robust Model Predictive Control". Automatica* 39, 1109-1112*.*

[5] *L. Zhang, J. Wang, K. Li* (2013)*: Min-Max MPC for LPV Systems Subject to Actuator Saturation by a Saturation-dependent Lyapunov Function. Proc. of the* 32th *Chinese Control Cnference, Xi'an, China,* 4087-4092*.*

[6] *J. Oravec, M. Bakošová* (2015)*: Alternative LMI-based Robust MPC Design Approaches. In Proceedings of the 8th IFAC Symposium on Robust Control Design, Elsevier, Bratislava, Slovak Republic,* 180-184*.*