Semidefinite Optimization

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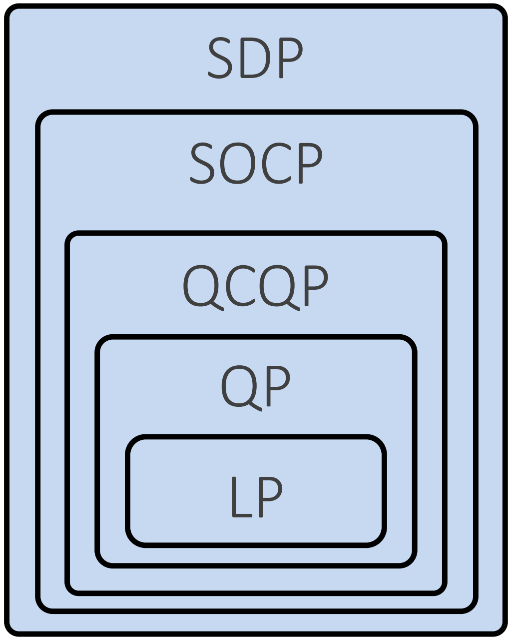
Semidefinite Optimization

– *super* class of convex optimization problems



Convex Optimization

Linear Programming

– *standard* LP:

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.}: \quad & Ax = b \\ & x \succeq 0 \end{aligned}$$

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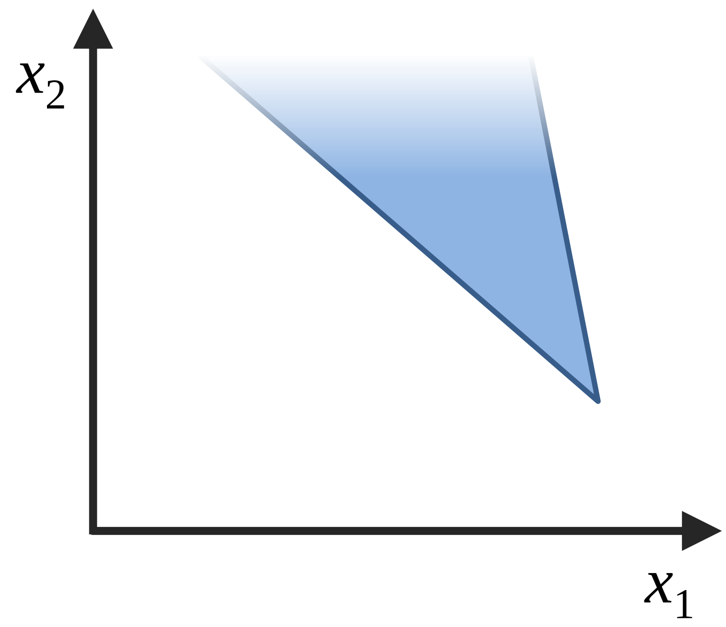
– feasibility set:

  – *polyhedron*



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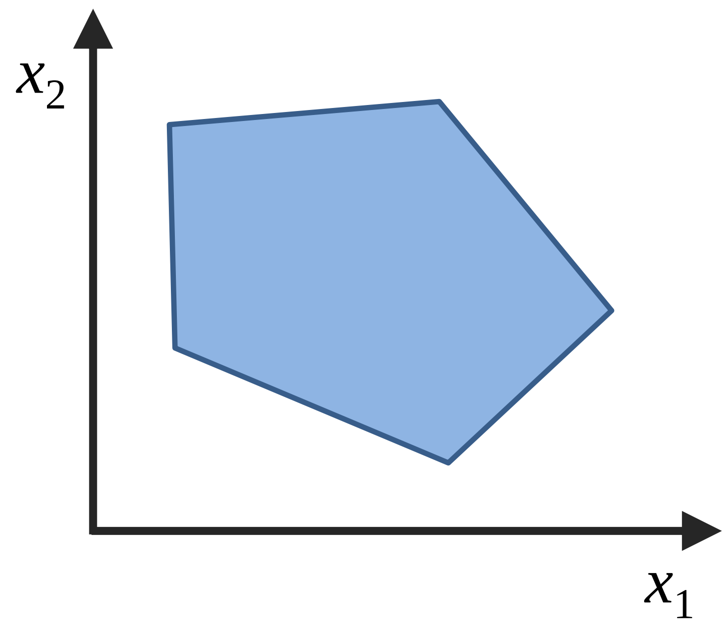
– *polytope*



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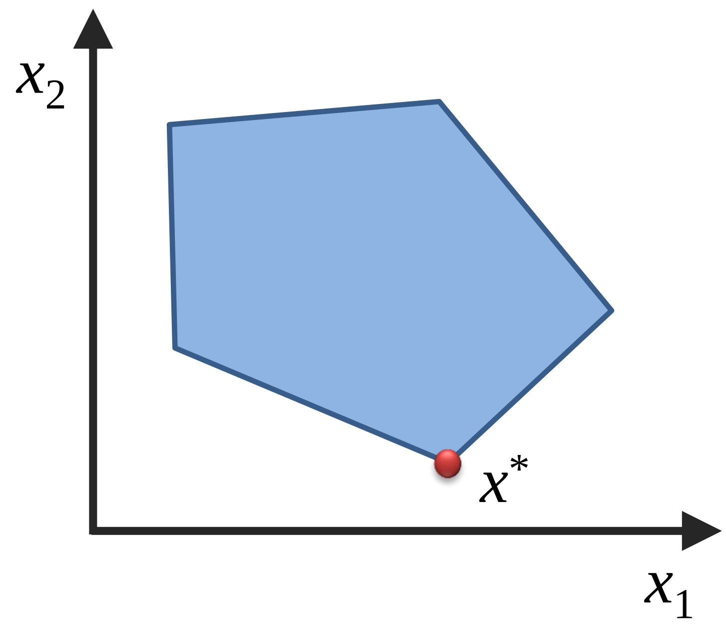

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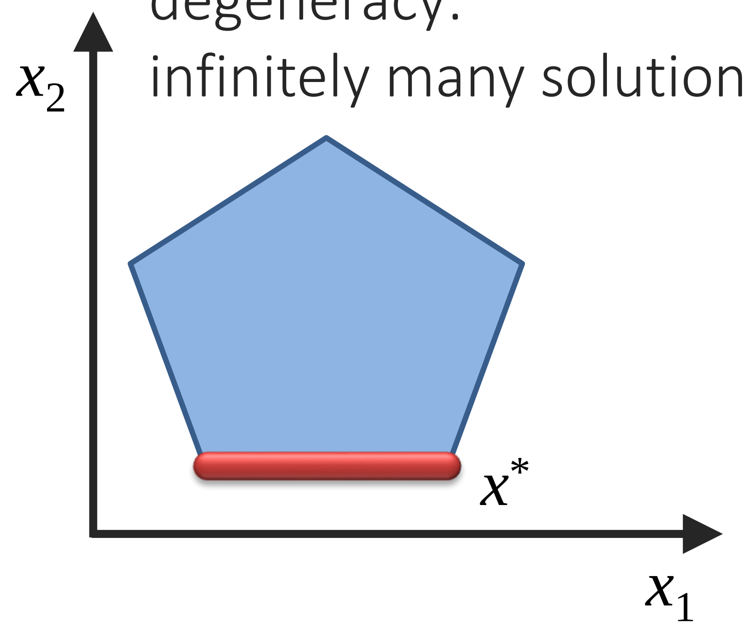
degeneracy:
infinitely many solutions



Linear Programming

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$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.}: \quad & Ax = b \\ & x \succeq 0 \end{aligned}$$

– feasibility set:

– *polyhedron:* $\mathbb{X}_{\text{feas}} = \text{convhull}(u_1, \ldots, u_r) + \text{conichull}(v_1, \ldots, v_s)$

– *polytope:* $\mathbb{X}_{\text{feas}} = \text{convhull}(u_1, \ldots, u_r)$

$$\text{convhull}(u_1, \ldots, u_r) = \{\sum_{i=1}^r \lambda_i u_i | \sum_{i=1}^r \lambda_i = 1, \lambda_i \succeq 0, i = 1, \ldots, r\}$$

$$\text{conichull}(v_1, \ldots, v_r) = \{\sum_{i=1}^s \lambda_i v_i | \lambda_i \succeq 0, i = 1, \ldots, s\}$$

Linear Programming

– *primal* LP:

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.}: \quad & Ax = b \\ & x \succeq 0 \end{aligned}$$

– *dual* LP:

$$\begin{aligned} \max \quad & b^\top y \\ \text{s.t.}: \quad & A^\top y \preceq c \end{aligned}$$

Semidefinite Programming

– optimization of a linear function subject to constraints of *linear matrix inequality*:

$$A_0 + \sum_{i=0}^{m} A_i X_i \succeq 0 \qquad\qquad A_i = A_i^\top \in \mathbb{S}^n$$

– decision variables are symmetric matrices:

$$X_i = X_i^\top \in \mathbb{S}^n$$

Semidefinite Programming

– *standard* SDP:

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{s.t.}: \quad & \langle A_i, X \rangle = b_i \\ & X \succeq 0 \end{aligned}$$

$$A_i, C, X \in \mathbb{S}^n, \quad \langle C, X \rangle = \operatorname{tr}\left(X^\top Y\right) = \sum_{i,j} C_{i,j} X_{i,j}$$

Semidefinite Programming

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– feasibility set:

– *spectrahedron*

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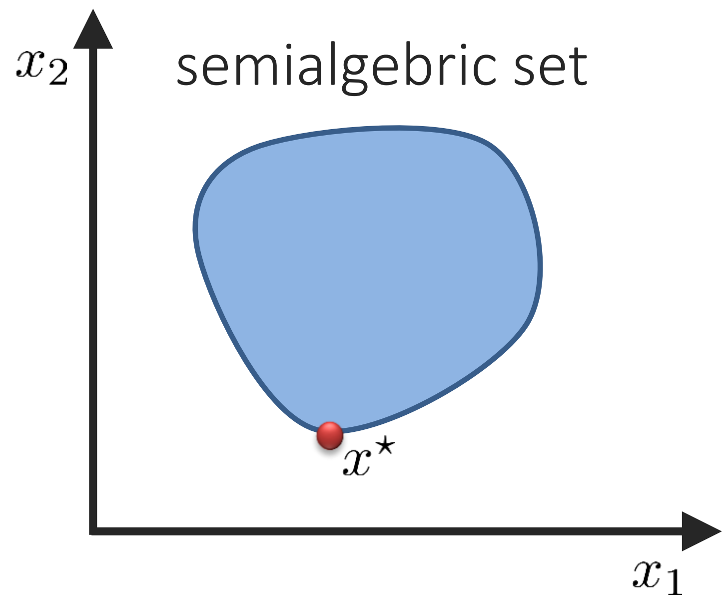

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– feasibility set:

– *spectrahedron*

$$\mathbb{X}_{\text{feas}} \left\{ x \in \mathbb{R}^n : A_0 + \sum_{i=0}^{m} A_i X_i \succeq 0 \right\}$$

Semidefinite Programming

– *primal* SDP:

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{s.t.}: \quad & \langle A_i, X \rangle = b_i \\ & X \succeq 0 \end{aligned}$$

$$A_i, C, X \in \mathbb{S}^n, \quad \langle C, X \rangle = \operatorname{tr}\left(X^\top Y\right) = \sum_{i,j} C_{i,j} X_{i,j}$$

– *dual* SDP:

$$\begin{aligned} \min \quad & b^\top y \\ \text{s.t.}: \quad & \sum_{i=1}^{m} A_i y_i \preceq C \end{aligned}$$

Lyapunov Stability

– Lyapunov function:

$$V(x) = 0 \quad \Leftarrow x = 0$$

$$V(x) > 0 \quad \Leftarrow x \neq 0$$

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} < 0$$

– quadratic Lyapunov function:

$$V : \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(x) = x^\top P x$$

– Lyapunov stability (continuous time domain)

$$A^\mathrm{T} P + PA \preceq 0$$

– Lyapunov stability (discrete time domain)

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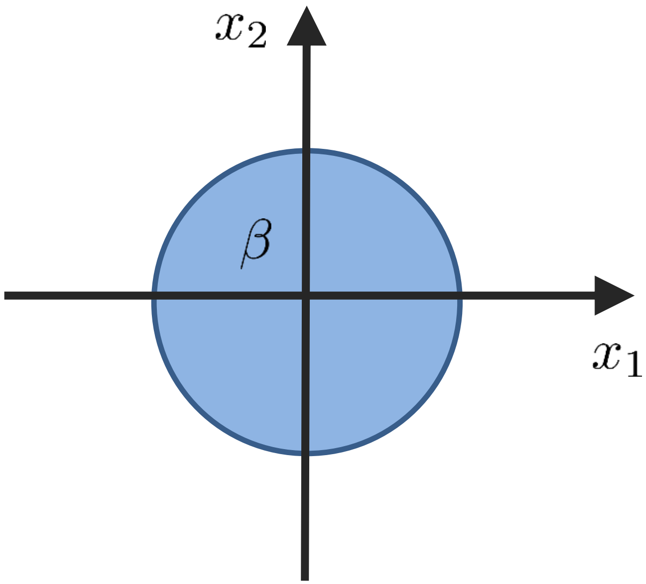
A. Lyapunov: The general problem of the stability of motion, 1892

Lyapunov Stability



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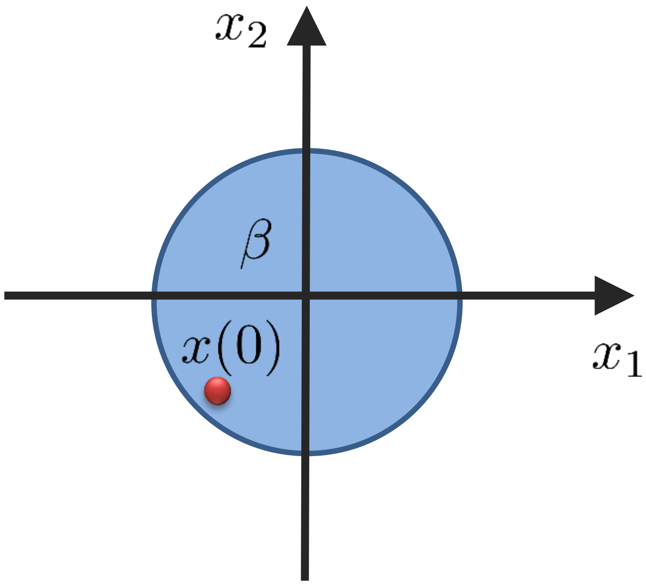

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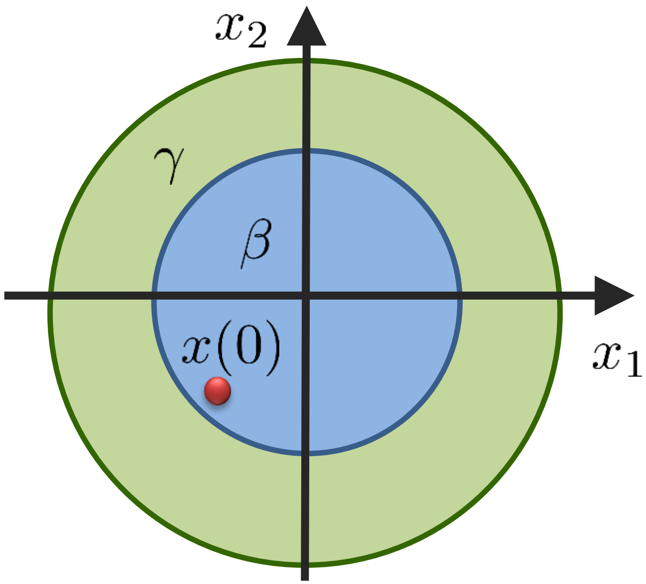

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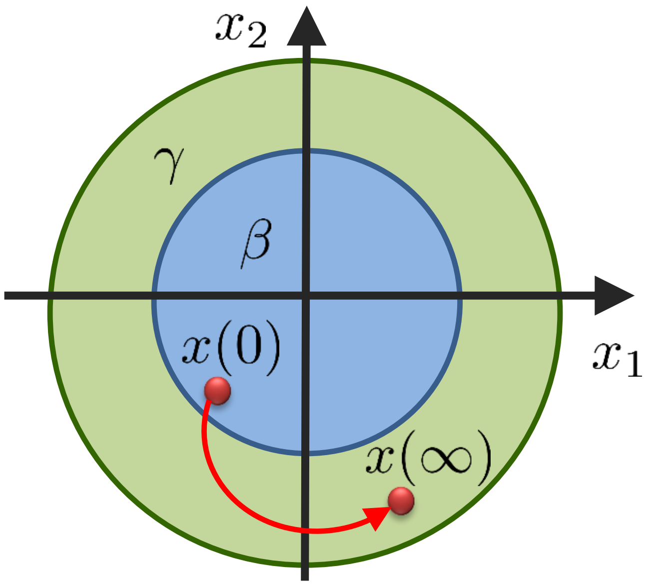

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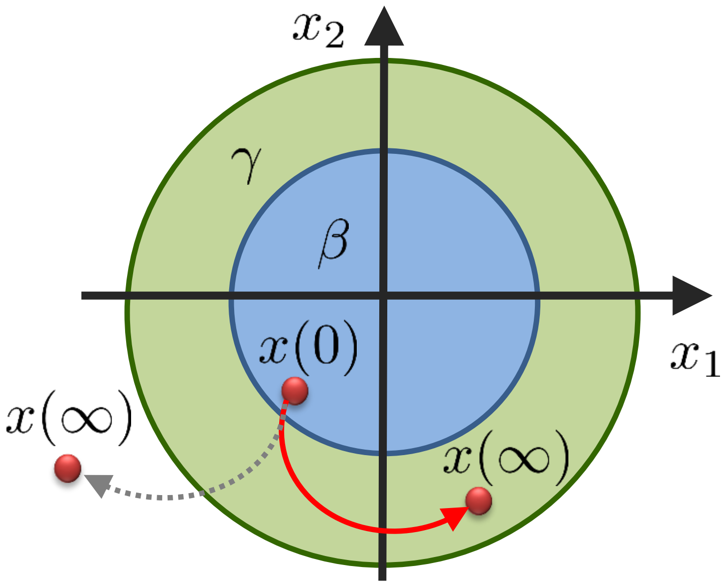

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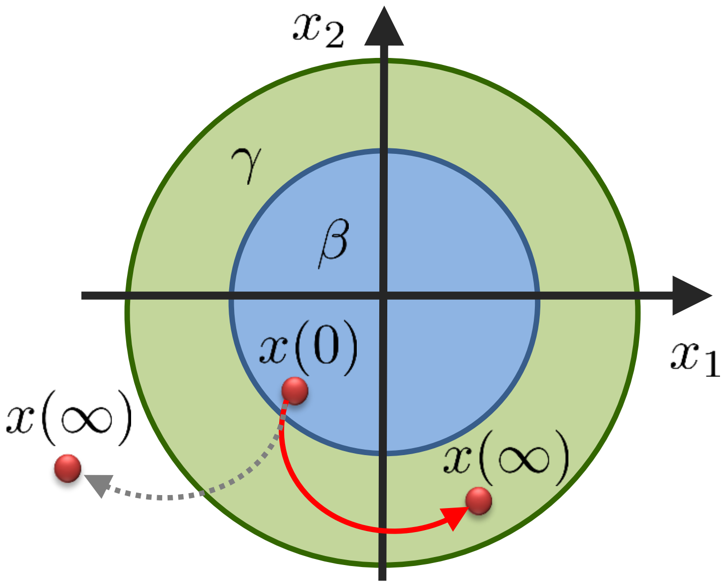

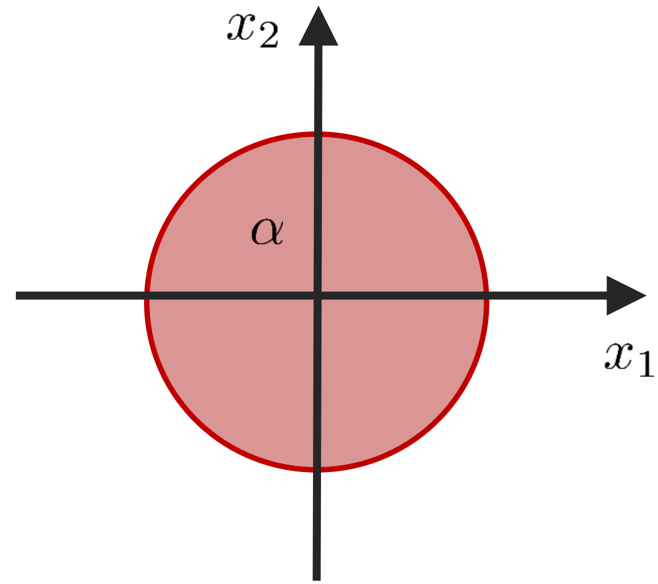

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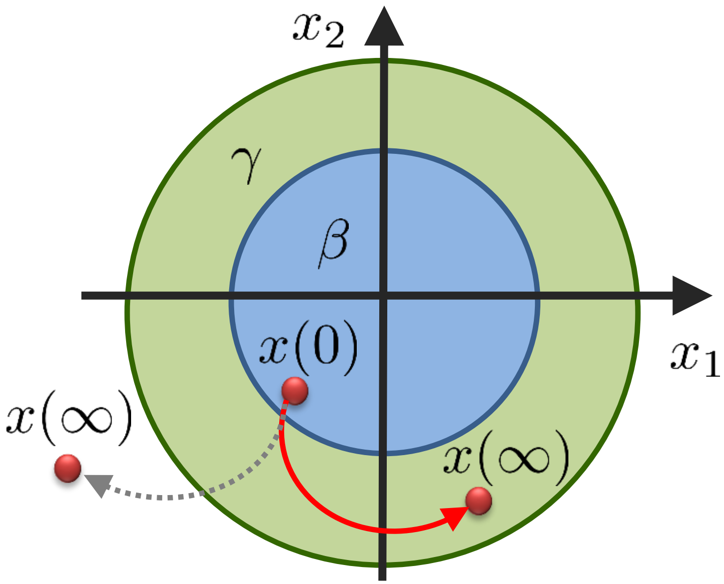

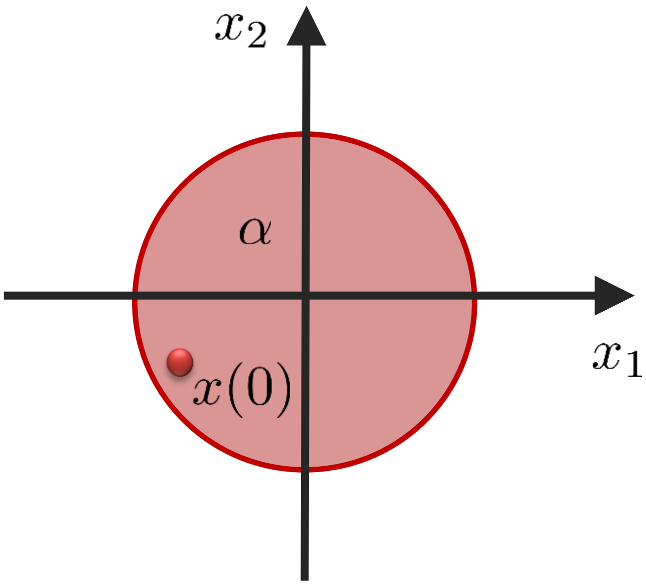

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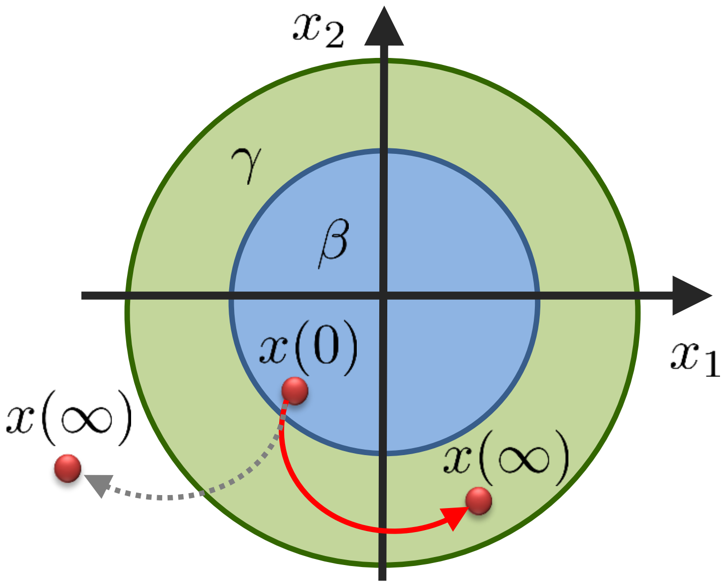

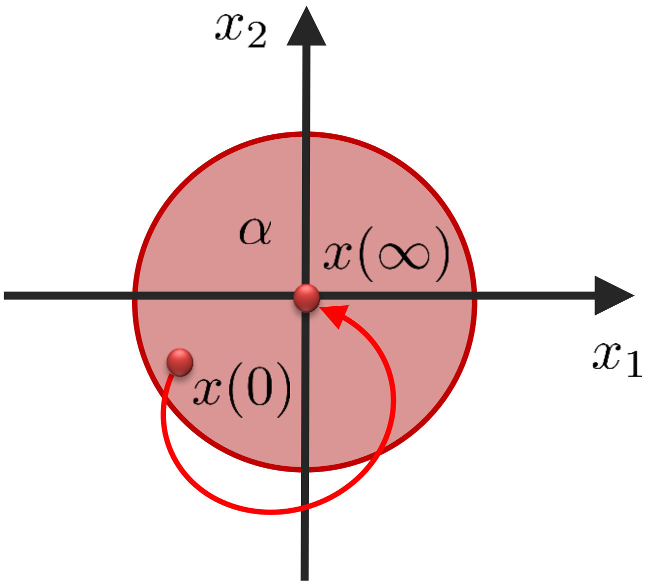

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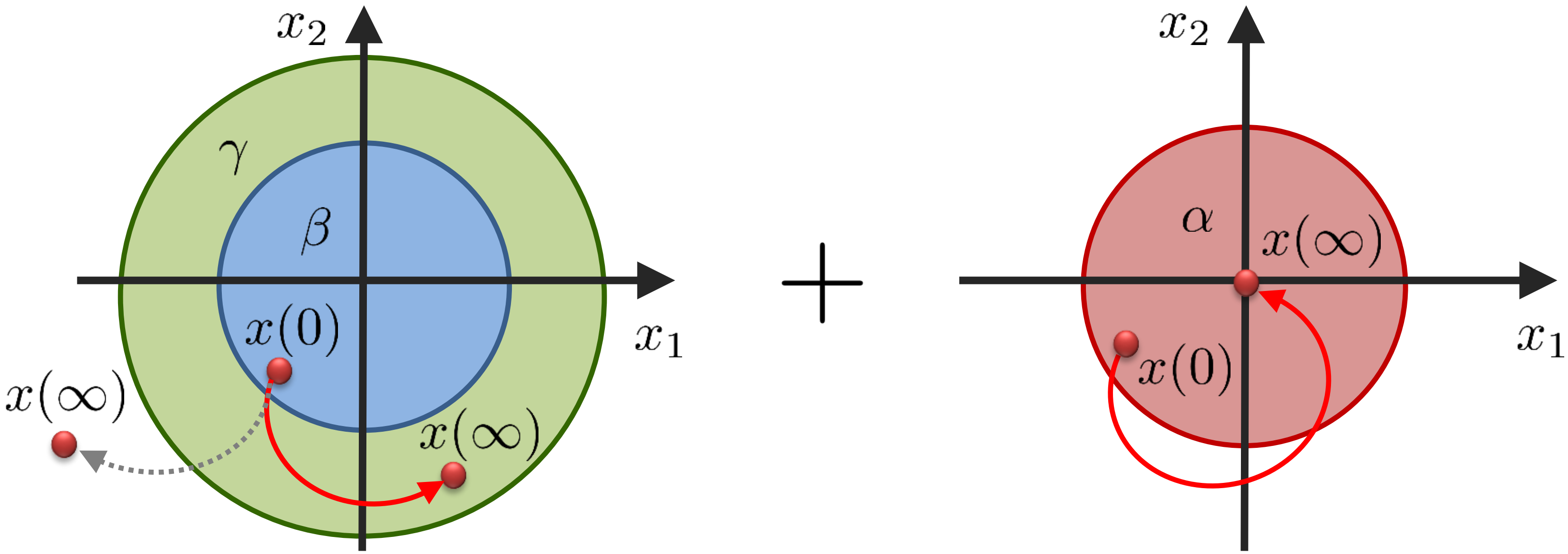

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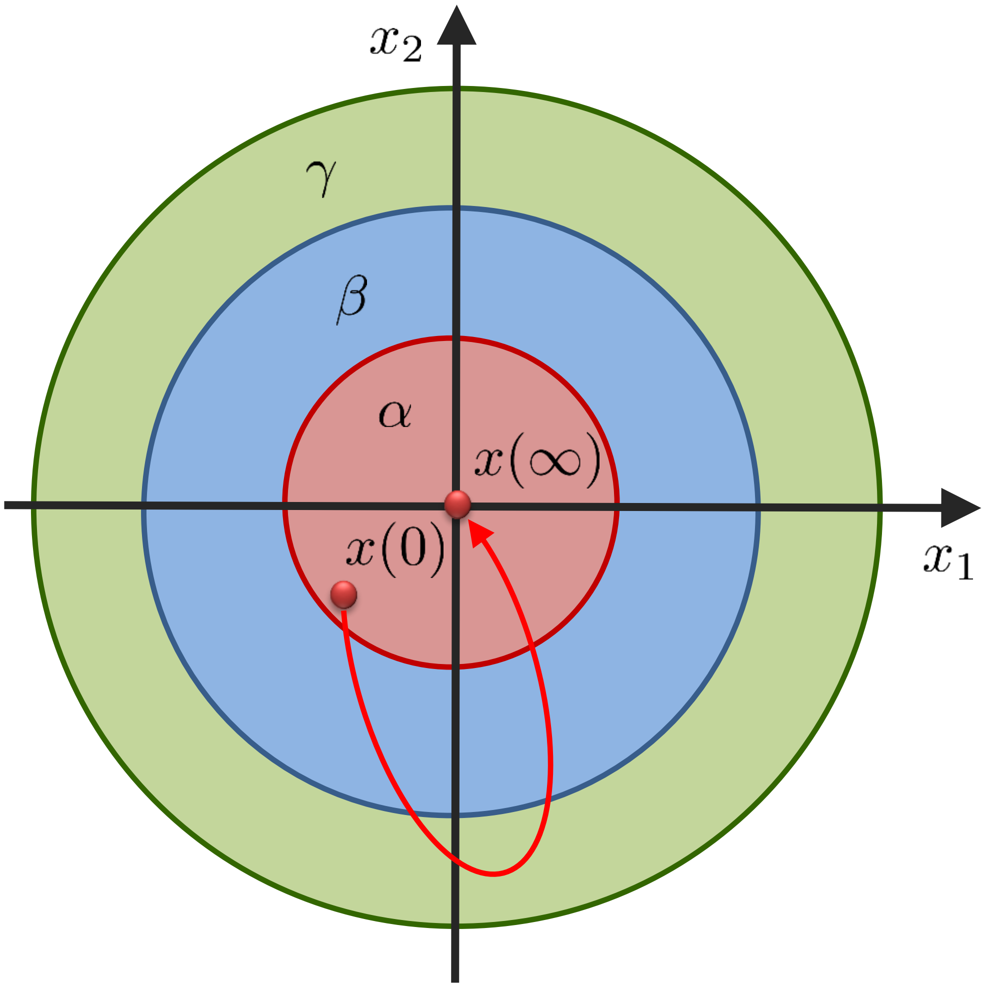

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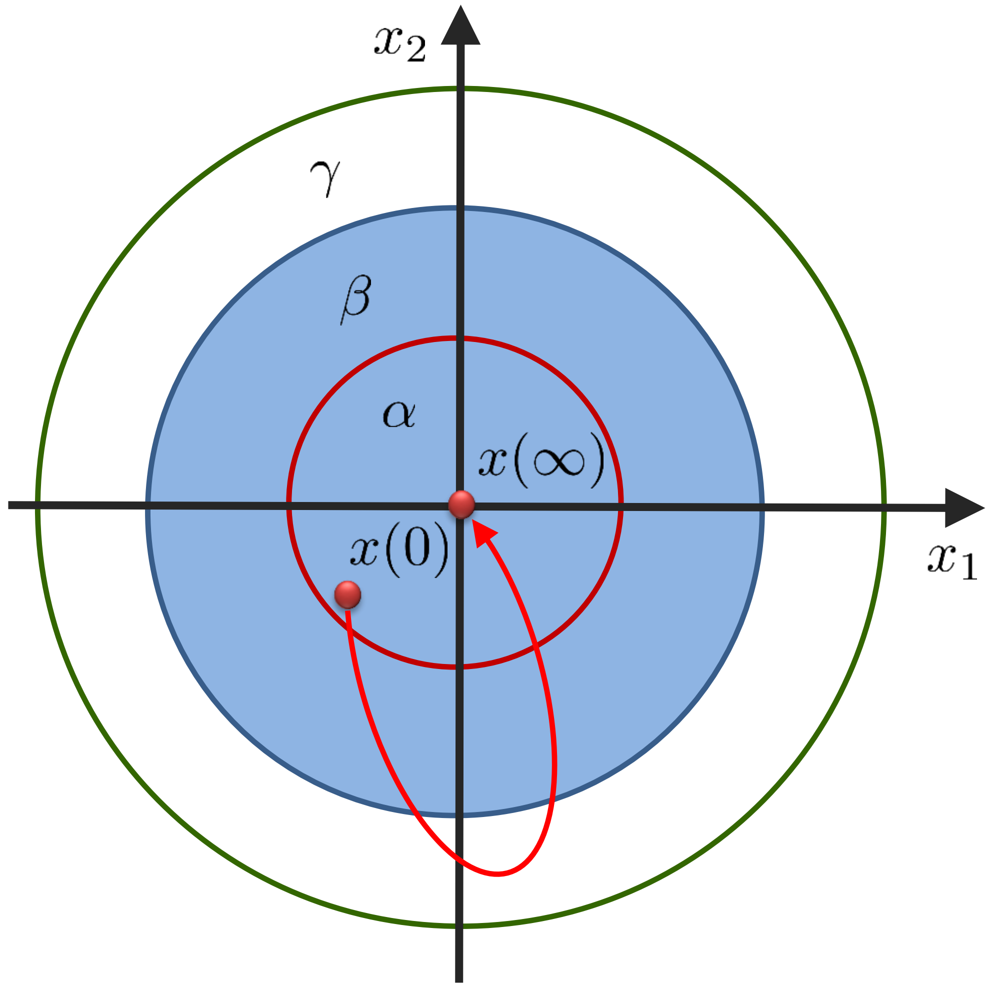

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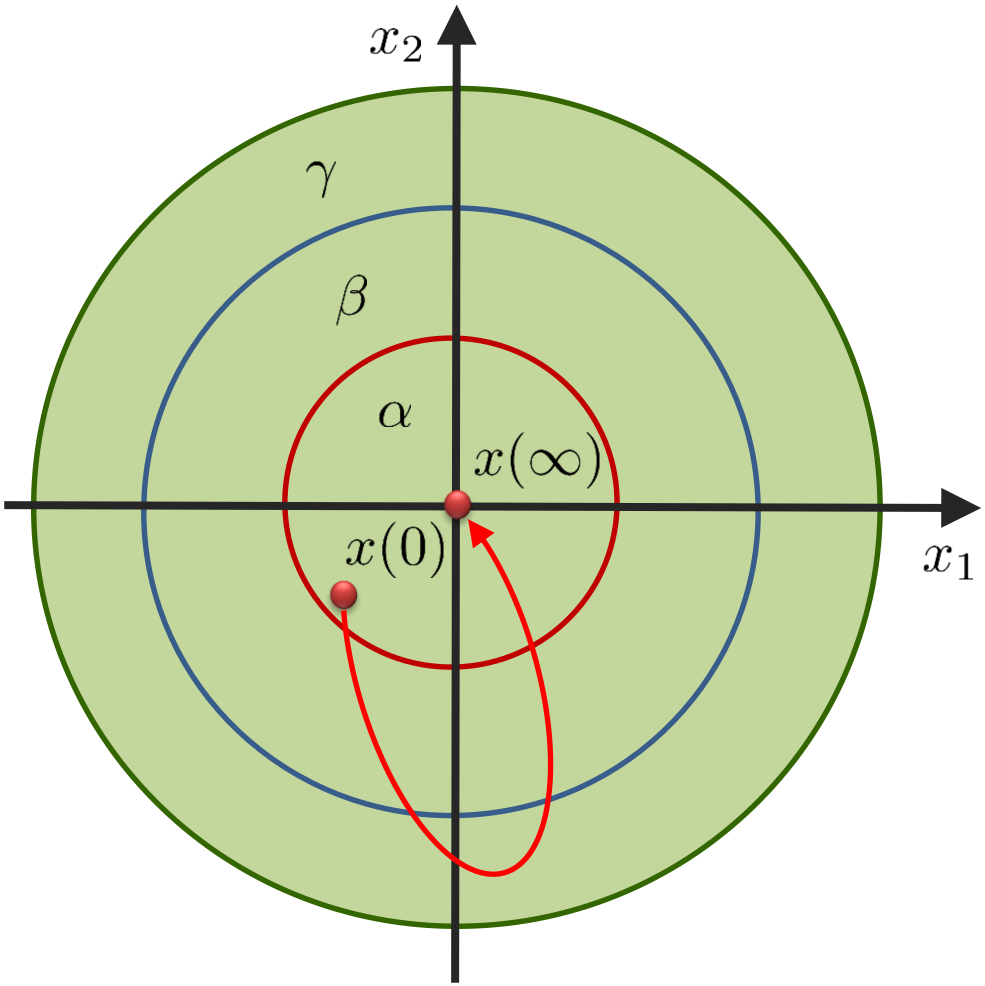

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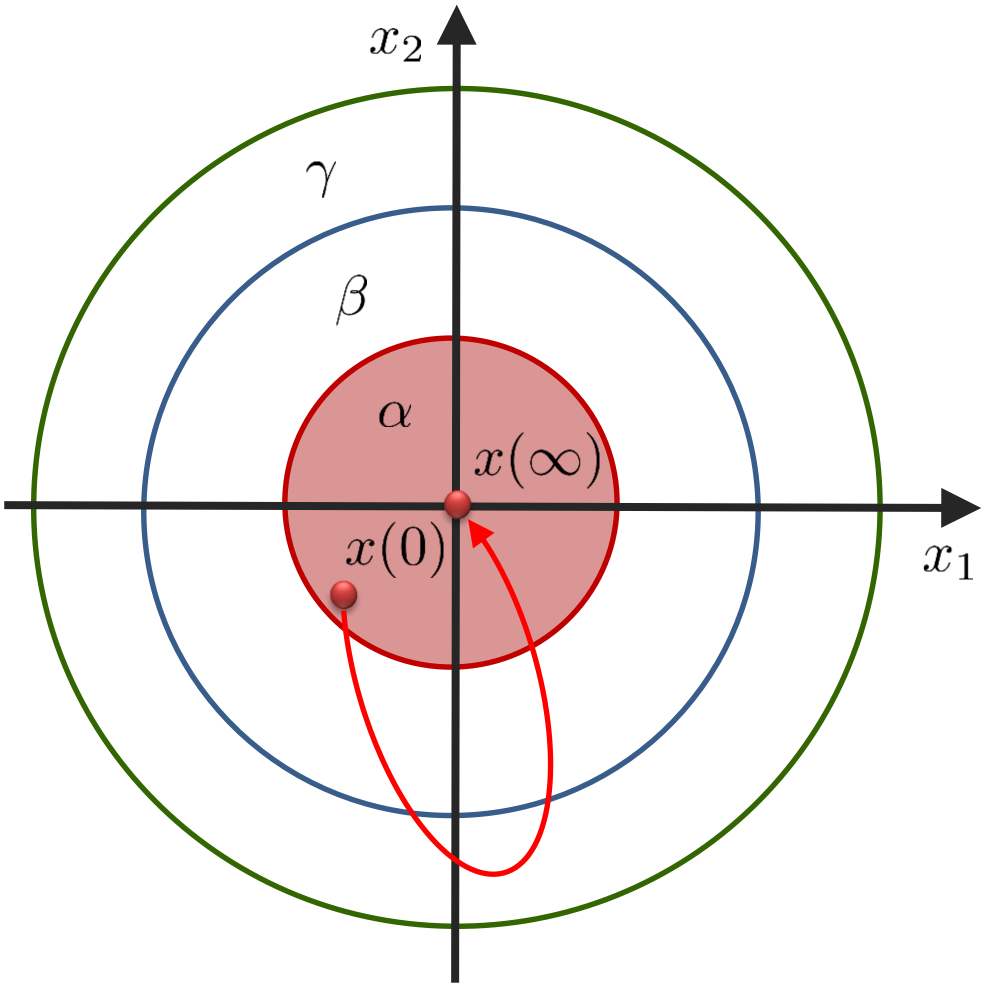

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YALMIP

– originally: *Yet Another LMI Parser*

– since 2003, by Johan Löfberg, Linköping University

– language for modeling and solution of the (non-)convex optimization problems in MATALB

– freely available: https://yalmip.github.io

– YALMIP : A Toolbox for Modeling and Optimization in MATLAB, In Proceedings of the CACSD Conference, Taipei, Taiwan, 2004.

YALMIP: How to

– define decision variable:

```
>>P = sdpvar(N,N,'symmetric') % symmetric matrix NxN
>>P = sdpvar(N,N)             % symmetric matrix NxN
>>P = sdpvar(N)               % symmetric matrix NxN
>>P = sdpvar(N,N,'full')    % non-symmetric matrix NxN
```

```
Linear matrix variable 2x2 (symmetric, real, 3 variables)
```

YALMIP: How to

– define constraints of optimization problem:

```
>>constr = [ P >= 0 ] % LMI

>>constr = constr + [[ A'*P*A - P ] <= 0] %LMI


++++++++++++++++++++++++++++++++

|    ID|              Constraint|

++++++++++++++++++++++++++++++++

|    #1|   Matrix inequality 2x2|

|    #2|   Matrix inequality 2x2|

++++++++++++++++++++++++++++++++
```

YALMIP: How to

– define objective of optimization problem:

```
>>obj = trace(P)


Linear scalar (real, 2 variables)
```

YALMIP: How to

– define settings of optimization problem:

```
>>opt = sdpsettings('solver','mosek') % SDP solver
>>opt = sdpsettings(opt,'verbose',0) % silent mode
```

YALMIP: How to

– solve optimization problem:

```
>>sol = optimize(constr,obj,opt)


yalmiptime: 0.5157
solvertime: 0.1963
      info: 'Successfully solved (MOSEK)'
   problem: 0
```

– parser time

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>>sol = optimize(constr,obj,opt)


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– optimization time

YALMIP: How to

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```
>>sol = optimize(constr,obj,opt)


yalmiptime: 0.5157
solvertime: 0.1963
      info: 'Successfully solved (MOSEK)'
   problem: 0
```

– no problems detected

YALMIP: How to

– solve optimization problem:

```
>>sol = optimize(constr,obj,opt)


yalmiptime: 0.5157

solvertime: 0.1963

      info: 'Infeasible problem (MOSEK)'

   problem: 1
```

– solution does not exist

YALMIP: How to

– solve optimization problem:

```
>>sol = optimize(constr,obj,opt)


yalmiptime: 0.5157

solvertime: 0.1963

      info: 'Unbounded objective function (MOSEK)'

   problem: 2
```

– do not trust the solution and bound the objective function

YALMIP: How to

– solve optimization problem:

```
>>sol = optimize(constr,obj,opt)


yalmiptime: 0.5157
solvertime: 0.1963
      info: 'Numerical problems (MOSEK)'
   problem: 4
```

– returned solution may not be correct – double-check feasibility

YALMIP: How to

– recover solution of optimization problem:

```
>>P_opt = value(P)


>>eigP = eig(P_opt)

>>M = [ A'*P_opt*A - P_opt ]

>>eigM = eig(M)
```

YALMIP and SDP solvers

– free: CSDP (OPTI Toolbox), DSDP, LOGDETPPA,
PENLAB (nonline SDPs), SDPA (high-precision),
SDPLR, SDPT3 (logdet), SDPNAL, SEDUMI (default)

– free for academia: MOSEK, PENSDP

– commercial: LMILAB (*not recommended*), PENBMI

Convex Optimization

Linear programming:

$$\min \; c^\top x$$
$$\text{s.t.}: \; Ax = b$$

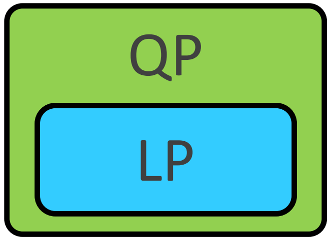
LP

CP

Convex Optimization

Quadratic programming:

$$\begin{aligned} \min \quad & x^\top Q x + c^\top x \\ \text{s.t.}: \quad & Ax = b \end{aligned}$$

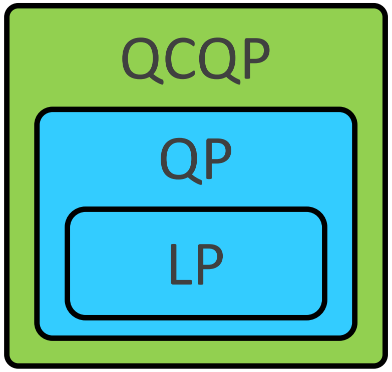


CP

Convex Optimization

Quadratically constrained quadratic programming:

$$\min \quad x^\top Q_0 x + 2c_0^\top x + r_0$$
$$\text{s.t.}: \quad x^\top Q_i x + 2c_i^\top x + r_i \leq 0$$

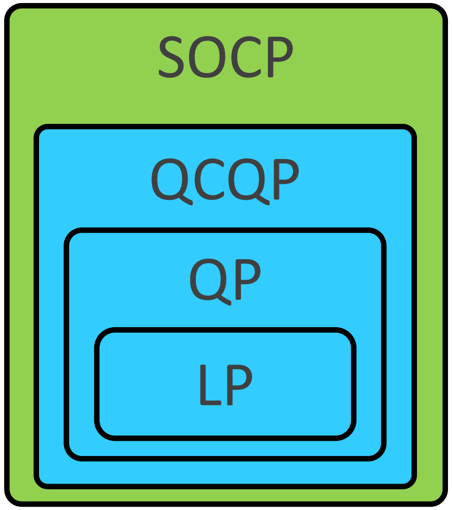


CP

Convex Optimization

Second-order cone programming:

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.}: \quad & Ax = b \\ & \|C_i x + d_i\| \le e_i^\top x + f_i \end{aligned}$$



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Second-order cone programming:

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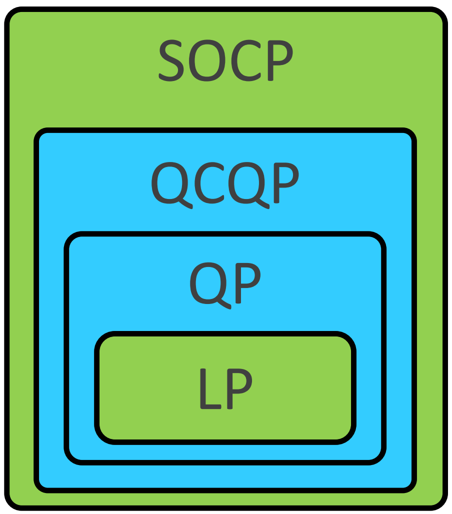

CP

LP:

$$\min \quad c^\top x$$
$$\text{s.t.}: \quad Ax = b$$
$$0 \leq e_i^\top x + f_i$$

Convex Optimization

Second-order cone programming:

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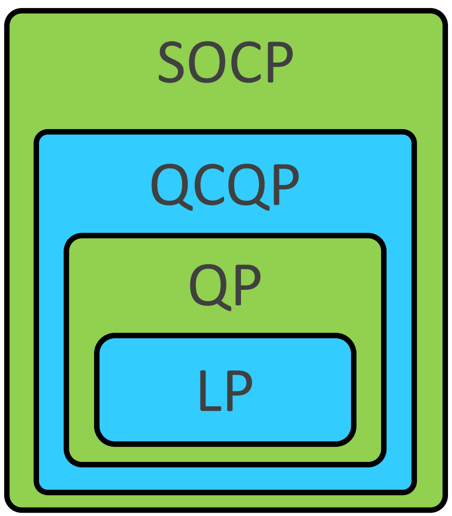

CP

$$\begin{aligned} \min \quad & t + c^\top x \\ \text{s.t.}: \quad & Ax = b \\ & t > x^\top Q^{\frac{1}{2}} Q^{\frac{1}{2}} x \end{aligned}$$

QP:

$$\begin{aligned} \min \quad & x^\top Q x + c^\top x \\ \text{s.t.}: \quad & Ax = b \end{aligned}$$

S. Boyd – L. Vandenberghe: Convex Optimization, 2004

Convex Optimization

Second-order cone programming:

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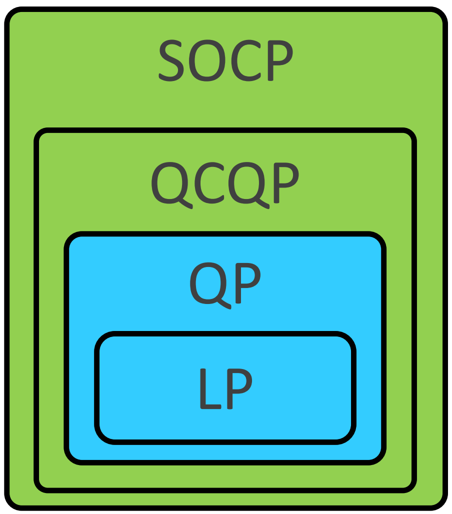

CP

$$\min \ t + 2c_0^\top x + r_0$$
$$\text{s.t.}: \ Ax = b, \ t > x^\top Q_0^{\frac{1}{2}} Q_0^{\frac{1}{2}} x$$
$$x^\top Q_i^{\frac{1}{2}} Q_i^{\frac{1}{2}} x \leq -2c_i^\top x - r_i$$

QCQP:

$$\min \ x^\top Q_0 x + 2c_0^\top x + r_0$$
$$\text{s.t.}: \ x^\top Q_i^{\frac{1}{2}} Q_i^{\frac{1}{2}} x + 2c_i^\top x + r_i \leq 0$$

Convex Optimization

Second-order cone programming:

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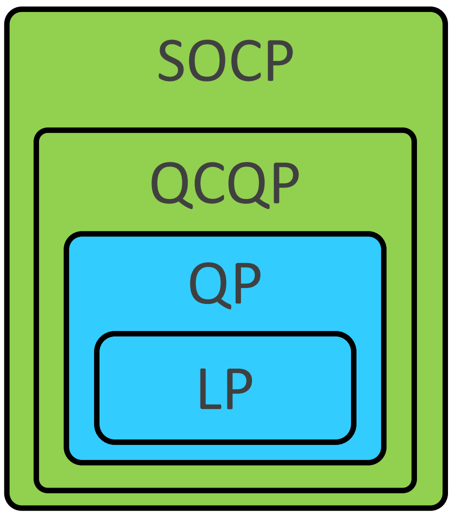

CP

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$$x^\top Q_i^{\frac{1}{2}} Q_i^{\frac{1}{2}} x \le -2c_i^\top x - r_i$$

QCQP:

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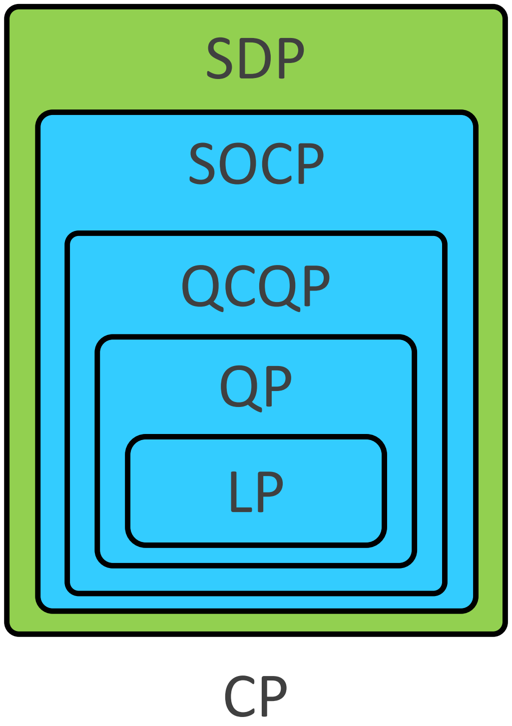
Convex Optimization

Semidefinite programming:

$$\min \; c^\top x$$
$$\text{s.t.}: \; Ax = b$$
$$F(x) = F_0 + \sum_{i=1}^{N} F_i x_i \succ 0$$



MALTAB/YALMIP solvers: MOSEK, PENBMI, CSDP, DSDP, SDPLR, SDPT3, SDPNAL, SEDUMI

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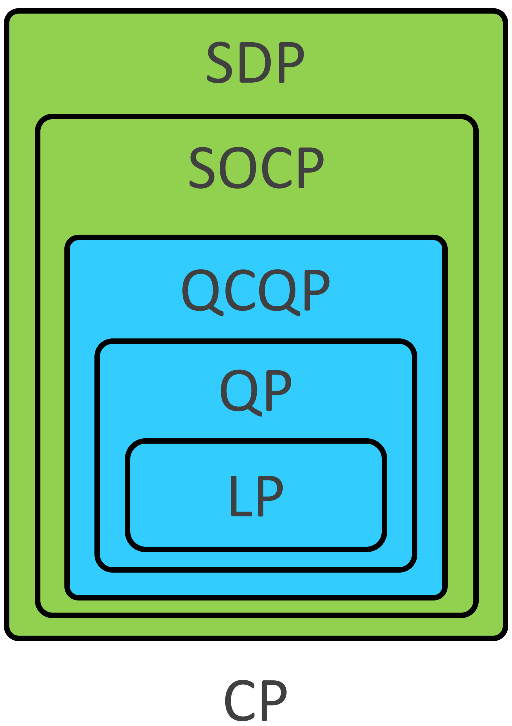

$$\min \; c^\top x$$
$$\text{s.t.}: \; \begin{bmatrix} (e_i^\top + d_i)I & (A_i x + b_i) \\ (A_i x + b_i)^\top & (e_i^\top + d_i) \end{bmatrix} \succeq 0$$

Schur Complement

SOCP:

$$\min \; c^\top x$$
$$\text{s.t.}: \; Ax = b$$
$$\|C_i x + d_i\| \leq e_i^\top x + f_i$$

References

– G. Blekherman , P. A. Parrilo and R. R. Thomas (2013): Semidefinite Optimization and Convex Algebraic Geometry

– S. Boyd, L. Vandenberghe (2004): Convex Optimization. Cambridge University Press.

– S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan (1994): Linear Matrix Inequalities in System and Control Theory. SIAM.

– J. Löfberg, YALMIP : A Toolbox for Modeling and Optimization in MATLAB, In Proceedings of the CACSD Conference, Taipei, Taiwan, 2004.